DOCUMENT RESUME

ED 173 152 SE 028 247

AUTHOR	Allen, Frank B.; And Others
TITLE	Mathematics for High School, Elementary Functions, (Part 2). Commentary For Teachers. Preliminary Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	60
NOTE	177p.; For related documents, see SE 028 246 and ED 135 630; Contains occasional light and broken type
EDRS PRICE	MF01/PC08 Plus Postage.
DESCRIPTORS	*Algebra; Curriculum; *Curriculum Guides; *Instruction; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Trigonometry
IDENTIFIERS	*Functions (Mathematics); *School Mathematics Study Group

ABSTRACT

This is part two of a two-part manual for teachers using SMSG high school text materials. Each chapter contains a commentary on the text, answers to the exercises, and a set of illustrative test questions. Chapter topics include exponential and logarithmic functions and circular functions. (MP)

SCHOOL
MATHEMATICS
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MATHEMATICS FOR HIGH SCHOOL

ELEMENTARY FUNCTIONS (Part 2)

COMMENTARY FOR TEACHERS

(preliminary edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

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ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

COMMENTARY FOR TEACHERS

for

~~ELE~~MENTARY FUNCTIONS

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Chapter 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Introduction.

The purpose of this chapter is to study the properties of the exponential functions $f : x \to a^x$ $(a > 0)$ and their inverses the logarithmic functions.

It is assumed that the student is familiar with the laws of exponents, in particular with

$$a^{r+s} = a^r a^s$$

and

$$(a^r)^s = a^{rs}$$

where r and s are rational numbers. Nevertheless, these matters are reviewed in the first two sections in connection with a concrete problem - the growth of a colony of bacteria. In Section 4-3, a^x is given a meaning when x is irrational. Another treatment of Sections 4-1 and 4-2 is given in the Appendix. This alternative introduces and solves the functional equation $f(x + y) = f(x) \cdot f(y)$. We believe that for exceptional students, this approach will be very illuminating.

Beginning with Section 4-4, the material will be new to the student. The method of linear approximation used in Chapter 2 is applied to exponential graphs. It is shown that if $f : x \to a^x$ the slope function is

$$f' : x \to ka^x$$

where c is the slope at $x = 0$.

Throughout the chapter emphasis is placed on the simplest non-trivial case, $a = 2$. We assume without proof that the graph of $x \rightarrow 2^x$, has a tangent at the point $P(0, 1)$. By joining P to nearby points, it is made plausible that the slope k of this tangent is approximately 0.69.

Once this is granted there is plain sailing. It is easy to show that if $f(x) = 2^x$, $f'(x) = k2^x$. By the device of expressing a as a power of 2 $(a = 2^\alpha)$ explained in 4-4, we show that for

$$f : x \rightarrow a^x$$

the slope function is

$$f' : x \rightarrow \alpha k a^x.$$

It is then natural to choose the base a so that $\alpha k = 1$ and hence so that $f' = f$. This leads to an intuitively simple way of introducing the base e and estimating its value.

In the appendix (4-12) there will be found a further discussion of e^x, in particular, some indication of the connection between equations (5) and (7) in 4-6.

Because of the emphasis put on the special case $a = 2$, it is important to familiarize the student very thoroughly with the use of the table of values of 2^x. Some remarks concerning the construction of this table are given below. The student should also become accustomed to the full-page graph of $y = 2^x$. A blackboard drawing of this graph would be helpful.

The applications discussed in 4-7 are of three types: to radioactive decay, to compound interest and to cooling. It is expected that at least the first type will be included in view of the current interest in radioactivity. It has the advantage that only powers of 2 need be involved.

Section 4-8 is a discussion of the important topic of inverse functions. Time spent on this section will be well repaid because of the light which it throws on the function concept. This material could have been introduced much earlier in the text (in Chapter 3 after composition of functions or even at the end of Chapter 1). However, it was felt that by placing the section in this chapter, the time lapse between idea and application is reduced to a minimum. The application of course is to logarithmic functions (4-9).

An effort has been made to treat logarithmic functions in a significant way by using the principal results of 4-8.

4-1.

It may be appropriate at this point to review the basic meanings of rational exponents and work a few exercises of the following kind:

1. Find a simpler name for

a) $2^3 \cdot 2^2$ b) $2^3 / 2^2$ c) $2^2 \cdot 2^3$ d) $4^{3/2}$

e) $4^{-3/2}$ f) $8^{2/3}$ g) $8^{-2/3}$ h) $(-8)^{-2/3}$

2. Find a simpler name for

a) $a^m \cdot a^n$ b) a^m / a^n c) $(a^m)^n$

d) a^{-n} e) $(ab)^m$ f) $(\frac{a}{b})^m$

g) What restrictions do we have on a, b, m, n? We avoid values that yield a zero denominator or an expression of the form 0^0.

Solutions to Exercises 4-1 Pages 220-229

1. Identity (1) says that if r and s are rational

$2^{r+s} = 2^r \cdot 2^s$ or

$$\frac{2^{r+s}}{2^r} = 2^s.$$

If we substitute "0" for "s", we get

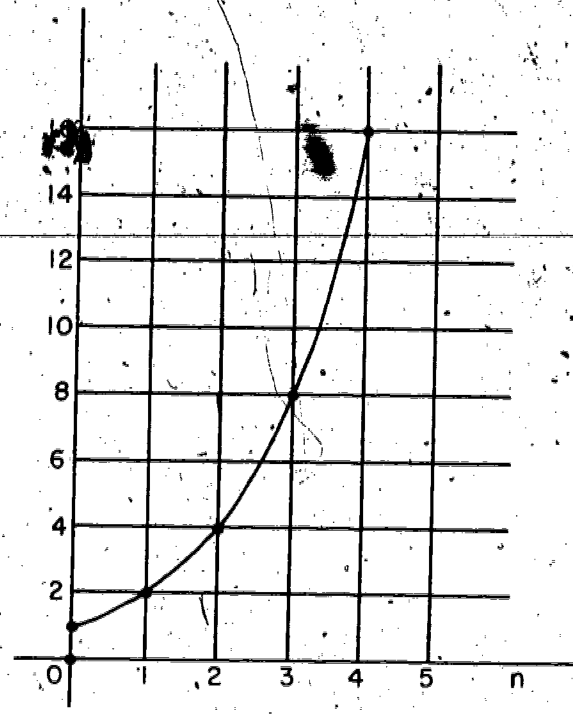
$$\frac{2^{r+0}}{2^r} = 2^0 \quad \text{or} \quad 2^0 = \frac{2^r}{2^r} = 1 .$$

2. As in Problem (1) if we substitute "- r" for "s", we get

$$2^{r+(-r)} = 2^r \cdot 2^{-r} \quad \text{or} \quad 2^{-r} = \frac{2^{r+(-r)}}{2^r} = \frac{2^0}{2^r} = \frac{1}{2^r}$$

3. Let $N(n) = 10^6(2^n)$.

n	$N(n)$ in millions
0	1
1	2
2	4
3	8
4	16



4. Let the number of bacteria after n days be represented by $N(n)$. We have then, $N(n) = 10^6(2^n)$. Thus,

$$\frac{N(a)}{N(b)} = \frac{10^6 \cdot 2^a}{10^6 \cdot 2^b} = 2^{a-b}$$

or $N(a) = N(b)2^{a-b}$ or if $a = b + k$, $N(b + k) = N(b)2^k$.

Hence $\dfrac{N(n+5)}{N(n+2)} = 2^{(n+5) - (n+2)} = 2^3 = 8$.

5. In this problem we seek

$$\frac{N(n+7)}{N(n+3)} = 2^{(n+7) - (n-3)} = 2^{10} = 1024.$$

6. Let $a = N(100) = 10^6 \cdot 2^{100}$. We seek an x such that $N(x) = \frac{a}{4}$, or $N(x) = a \cdot 2^{-2}$, or $N(x) = 10^6 \cdot 2^{100} \cdot 2^{-2}$ $= 10^6 \cdot 2^{98} = N(98)$. Therefore $x = 98$.

7. This problem involves the more general situation where $N(n) = N(0)b^n$. As in Problem 4,

$$\frac{N(a+k)}{N(a)} = \frac{N(0)b^{a+k}}{N(0)b^a} = b^k$$

or $N(a+k) = N(a)b^k$. Let $a = 3$, $k = 3/2$. Then $N(3 + 3/2) = 1,600,000 = N(3)b^{3/2} = 200,000\ b^{3/2}$. Therefore, $1,600,000 = 200,000\ b^{3/2}$ or $b^{3/2} = 8$ and $b = 4$. Hence, $N(n) = N(0)4^n$ and in particular $N(3) = 200,000 = N(0)4^3$. Therefore, $N(0) = 200,000 \cdot 4^{-3}$.

The formula for $N(n)$ now may be written as

$$N(n) = 200,000 \cdot 4^{n-3}$$

(a) If $n = 5$, $N(5) = 200,000 \cdot 4^{5-3} = 3,200,000$

(b) If $n = 1\ 1/2$, $N(1\ 1/2) = 200,000 \cdot 4^{1\ 1/2\ -\ 3}$ $= 200,000 \cdot 4^{-3/2} = 200,000 \cdot 1/8 = 25,000$.

(c) If $N(n) = 800,000$, then $200,000 \cdot 4^{n-3}$ $= 800,000$ and $4^{n-3} = 4^1$. Therefore, $n - 3 = 1$ and $n = 4$.

Solutions to Exercises 4-2a. Pages 233-234

1. a) If both m and n are positive integers $a^m \cdot a^n$

$$\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ \text{factors}} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ \text{factors}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m+n\ \text{factors}} = a^{m+n}$$

If $m = 0$, then we know that $a^0 = 1$. Hence,

$a^m \cdot a^n = a^0 \cdot a^n = 1 \cdot a^n = a^n = a^{0+n} = a^{m+n}$. A similar argument holds if $n = 0$. Thus, the rule is established for $m, n = 0, 1, 2, \ldots$

b) Now suppose $n < 0$, $m \geq 0$. Then

$$a^m \cdot a^n = a^m \cdot \frac{1}{a^{-n}} = \frac{a^m}{a^{-n}} .$$

(1) If $m \geq -n$, let $m = -n + k$ where $k = 0, 1, 2, \ldots$ Then

$$\frac{a^m}{a^{-n}} = \frac{a^{-n+k}}{a^{-n}} = \frac{a^{-n} \cdot a^k}{a^{-n}} \qquad \text{from (2)}$$

$$= a^k = a^{m+n} , \quad \text{so} \quad a^m \cdot a^n = a^{m+n}$$

for this case.

(2) If $m < -n$ (m need not be 0), then let $m + k = -n$ where $k = 1, 2, 3, \ldots$. Hence,

$$\frac{a^m}{a^{-n}} = \frac{a^m}{a^{m+k}} \qquad \frac{a^m}{a^m \cdot a^k} = \frac{1}{a^k} = a^{-k} = a^{m+n} .$$

Once again $a^m \cdot a^n = a^{m+n}$, completing all the possibilities.

2. a) Suppose $m, n = 0, 1, 2, 3, \ldots$.

If $m = 0$, then $(a^m)^n = (a^0)^n = 1^n = 1 = a^0$

$$= a^{0 \cdot n} = a^{mn}.$$

If $n = 0$, then $(a^m)^n = (a^m)^0 = 1 = a^0 = a^{0 \cdot m} = a^{mn}$.

Now suppose $m \neq 0$, $n \neq 0$. Then $(a^m)^n$ means that a^m is used as a factor n times:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \ldots \cdot a^m}_{n \text{ factors of } a^m} .$$

But a^m itself is the product when a is taken as a factor m times, $a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$. Replacing "$a^m$" by its expanded form, "$a \cdot a \cdot \ldots \cdot a$" in $(a^m)^n$ we see that a is used as a factor mn times so that $(a^m)^n = a^{mn}$.

We have established the rule $(a^m)^n = a^{mn}$ for $m, n = 0, 1, 2, 3, \ldots$. In fact, if m or $n = 0$, the other need not be restricted to the non-negative integers. The proof goes through for this more general case.

b) Now suppose $m = -1, -2, \ldots; \ n = 1, 2, 3, \ldots$

Then $(a^m)^n = \left(\frac{1}{a^{-m}}\right)^n = \underbrace{\frac{1}{a^{-m}} \cdot \frac{1}{a^{-m}} \cdot \ldots \cdot \frac{1}{a^{-m}}}_{n \text{ factors}}$

$$= \frac{1}{(a^{-m})^n} = \frac{1}{a^{(-m)n}} = \frac{1}{a^{-mn}} = a^{mn}.$$

c) Now suppose, $m = 1, 2, \ldots; \ n = -1, -2, -3, \ldots$ then

$$(a^m)^n = \frac{1}{(a^m)^{-n}} = \frac{1}{a^{m(-n)}} = \frac{1}{a^{-mn}} = a^{mn}.$$

d) Finally, suppose $m = -1, -2, \ldots; \ n = -1, -2, \ldots$

then $(a^m)^n = \left(\frac{1}{a^{-m}}\right)^n = \frac{1}{\left(\frac{1}{a^{-m}}\right)^{-n}} = \frac{1}{\frac{1}{(a^{-m})^{-n}}}$

$$= \frac{1}{\frac{1}{a^{(-m)(-n)}}} = \frac{1}{\frac{1}{a^{mn}}} = a^{mn}.$$

This completes the proof that $(a^m)^n = a^{mn}$ for integral values of m and n.

3. $1000(8^{-2/3}) = 1000 \cdot \frac{1}{8^{2/3}} = 1000 \cdot \frac{1}{2^2} = 250$

$3(\frac{9}{4})^{-3/2} = 3(\frac{3}{2})^{-3} = 3(\frac{2}{3})^3 = 3 \cdot \frac{8}{27} = \frac{8}{9}$

4. $(4^{5/2})(8^{-1}) = (2^2)^{5/2}(2^3)^{-1} = 2^5 \cdot 2^{-3} = 2^2$

$(\frac{1}{2})^{-4/3} = (2^{-1})^{-4/3} = 2^{4/3} = 2^{1\ 1/3}$

$(2^{-2/9})^9 = 2^{-2}$.

As $x \to 2^x$ is an increasing function as x increases, in order of decreasing value from the left we have

$2^2,\ 2^{4/3},\ 2^{2/3},\ 2^{-2},\ 2^{-3}$ or

$(4^{5/2})(8^{-1}),\ (\frac{1}{2})^{-4/3},\ 2^{2/3},\ (2^{-2/9})^9,\ 2^{-3}$.

5. $2^{2.7} = 2^2 \cdot 2^{.7} = 4 \cdot 2^{7/10} = 4 \sqrt[10]{2^7} = 4 \sqrt[10]{128}$

6. a) If $8^m = (2^3)^2$, then $8^m = 8^2$ and $m = 2$.

b) If $8m = 2^{(3^2)}$, then $2^3 m = 2^9$ and

$m = 2^{9-3} = 2^6 = 64$.

7. a) If $2^{(4^5)} = 16^m$, then $2^{(4^5)} = (2^4)^m = 2^{4m}$ and

$4^5 = 4m$ so that $m = 4^4 = 256$.

b) If $(2^4)^5 = 16^m$ then $2^{20} = 2^{4m}$ and $20 = 4m$ so that $m = 5$.

8. $\frac{2^{b+c} - 2^b}{c} = \frac{2^b \cdot 2^c - 2^b}{c} = \frac{2^b(2^c - 1)}{c}$

It might be well to obtain some rational powers of 2 before resorting to the table on page 235. The table then becomes more meaningful to the student, and moreover, there is a better notion of the increasing nature of $f : x \rightarrow 2^x$. The rational powers of 2 that are most readily computed are $2^{.5}$, $2^{.25}$, $2^{.75}$, $2^{.125}$, $2^{.625}$, $2^{.875}$ which are obtained by successive square roots of 2 and taking appropriate products. Thus, $2^{.75} = 2^{.5} \cdot 2^{.25}$. This could have been obtained, also, by finding the square root of $\sqrt{8}$. Thus $2^{3/4} = 8^{1/4}$. If these values are obtained and listed in order, we get, approximately,

r	2^r
0.000	1.000
.125	1.091
.250	1.189
.375	1.296
.500	1.414
.625	1.542
.750	1.682
.875	1.834
1.000	2.000

Intermediate values of 2^r may be obtained by linear interpolation within a very small error. Thus, using the familiar interpolation technique we get from this table

$2^{.2} = 1.150$ instead of 1.149

$2^{.8} = 1.742$ instead of 1.741.

It might be helpful and a time-saver to have a class-chart of the table of values of 2^r as given on page 235.

Solutions to Exercises 4-2b. Pages 235-236

1. a) $2^{5/4} = 2^{1.25} = 2 \cdot 2^{.25} \approx 2(1.189) = 2.378$

   b) $2^{5/4} = 2\sqrt{\sqrt{2}} \approx 2 \cdot \sqrt{1.414} \approx 2(1.189) = 2.378$

2. a) $2^{1.15} = 2 \cdot 2^{.15} \approx 2(1.110) = 2.220$

   b) $2^{2.65} = 2^{2} \cdot 2^{.65} \approx 4(1.569) = 6.276$

   c) $2^{0.58} = 2^{.55} \cdot 2^{.03} \approx (1.464)(1.102101) \approx 1.49$

   d) $2^{-0.72} = 2^{-1+.28} = \frac{1}{2}(2^{.25})(2^{.03})$

   $$\approx .5 \cdot 1.189 \cdot 1.02101 \approx .61$$

3. a) $8^{.84} = (2^{3})^{.84} = 2^{2.52} = 2^{2} \cdot 2^{.5} \cdot 2^{.02}$

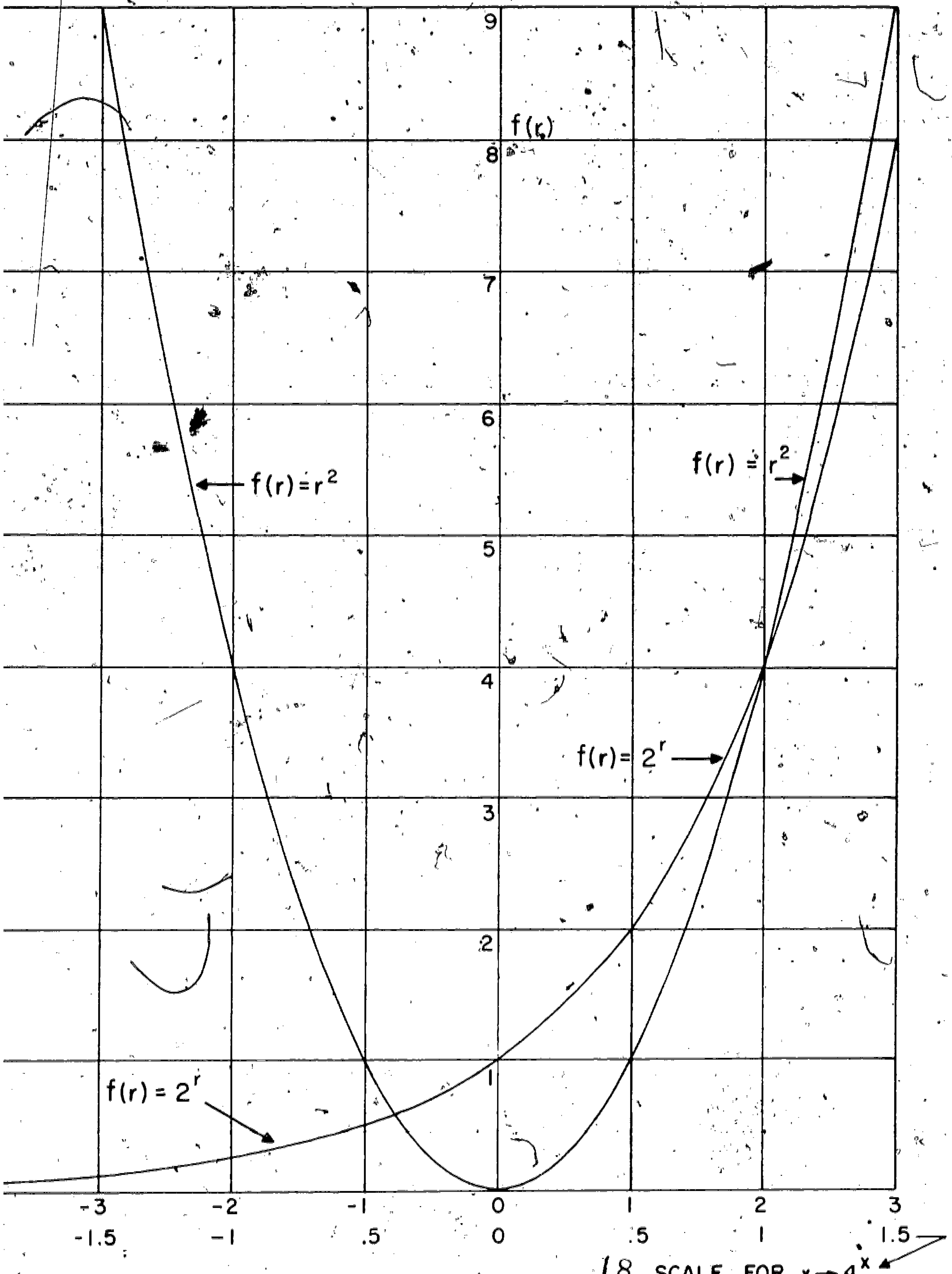
   $$\approx 4(1.414)(1.01396) \approx 5.74$$

   b) $.25^{-0.63} = (2^{-2})^{-0.63} = 2^{1.26} = 2^{1} \cdot 2^{.25} \cdot 2^{.01}$

   $$\approx 2(1.189)(1.00695) \approx 2.39.$$

4.

r	2^{r}
-4.0	.0625
-2.6	.0825
-3.2	.109
-2.8	.144
-2.4	.190
-2.0	.250
-1.6	.330
-1.2	.435
-0.8	.575
-0.4	.758
1.4	2.640
1.8	3.482
2.2	4.596
2.6	6.064
3.0	8.000


f(r)
9
8
7
6
5
4
3
2
1
$f(r) = r^2$
$f(r) = r^2$
$f(r) = 2^r$
$f(r) = 2^r$
-3 -2 -1 0 1 2 3
-1.5 -1 .5 0 .5 1 1.5
SCALE FOR $x \rightarrow 4^x$

Solutions to Exercises 4-3 Page 243

1. a) $2^{1.15} \approx 2.22 \approx 2.2$. We might permit as much as a
   b) $2^{2.65} \approx 6.28 \approx 6.3$ .2 leeway from these answers
   c) $2^{.58} \approx 1.49 \approx 1.5$ if the graph is not blown up.
   d) $2^{-0.72} \approx .61 \approx .6$

2. A comparison should reveal a difference of not more than .2.

3. a) $2^{\sqrt{3}} \approx 2^{1.73} \approx 3.32 \approx 3.3$
   b) $2^{\pi} \approx 2^{3.14} \approx 8.82 \approx 8.8$
   c) $2^{-\pi/4} \approx 2^{-.79} \approx .58 \approx .6$

4. No. If x is a real number such that $2^x = 0$, and if y is any real number whatever, then $y - x$ is a real number, 2^{y-x} is therefore defined, and we have

$$2^{y-x} \cdot 2^x = 2^{y-x} \cdot 0 = 0,$$

but $$2^{y-x} \cdot 2^x = 2^y ,$$

therefore $$2^y = 0$$

for every real number y. But $2^1 = 2$, so this is impossible.

5. a) If $2^x = 6$, $x \approx 2.58 \approx 2.6$
   b) If $2^x = .4$, $x \approx -1.32 \approx -1.3$
   c) If $2^x = 3.8$, $x \approx 1.92 \approx 1.9$

Solutions to Exercises 4-4 Pages 246-247

It is suggested that these exercises be checked by using the graph of $x \rightarrow 2^x$.

1. $3.4 \doteq 2 \cdot 1.7 = 2^1 \cdot 2^{.77} = 2^{\alpha}$, therefore,

   $\alpha = 1.77 \approx 1.8$

2. $2.64 = 2 \cdot 1.32 \approx 2^1 \cdot 2^{.40} = 2^{1.40} = 2^{\alpha}$, hence,

   $\alpha \approx 1.40 \doteq 1.4$

   $(2.64)^{0.8} \approx (2^{1.4})^{0.3} \doteq 2^{.42} \doteq 2^{.4} \cdot 2^{.02} \approx (1.320)(1.01396)$

   $\approx 1.34 \approx 1.3$.

   Using the graph of $x \to 2^x$ one obtains $\alpha \approx 1.4$ quickly. Then $2^{.42}$ is also read off the graph as 1.3.

3. $6.276 = 4 \cdot 1.569 \approx 2^2 \cdot 2^{.65} = 2^{2.65}$; $\alpha \approx 2.65$

   $6.276^{-0.6} \approx (2^{2.65})^{-0.6} = 2^{-1.59} = 2^{-2+.410} = 2^{-2} \cdot 2^{.41}$

   $\approx \frac{1}{4}(2^{.4} \cdot 2^{.01}) \approx \frac{1}{4}(1.320)(1.00695)$

   $\approx .33 \approx .3$

4. $5.2 = 4 \cdot 1.3 \approx 2^2 \cdot 2^{.38} = 2^{2.38}$

   $5.2^{2.6} \approx (2^{2.38})^{2.6} = 2^{6.188} = 2^6 \cdot 2^{.188}$

   $\approx 2^6 \cdot 2^{.15} \cdot 2^{.04}$

   $\approx 64(1.110)(1.02819) \approx 73$

4-5 Tangent Lines to Exponential Curves

It might be appropriate at this point to review the methods for finding the slope of a line, and the equation of a line. The slope, m, of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{provided } x_1 \neq x_2 .$$

The equation of a line through P_1 and P_2 is $y - y_1 = m(x - x_1)$ where m is the slope. See Section 2-1.

The following exercises might help in the recall:

I. Find the slope of the line P_1P_2 if P_1 and P_2 are located at

a) (1, 2), (4, 7)
b) (0, 1), (1, 3)
c) (0, 1), (.5, 1.3)
d) (0, -1), (.5, 1.3)
e) (-1, 2), (3, 5)
f) (-1, -2), (3, 5)
g) (-1, -2), (-3, -5)
h) (0, 0), (-3, -5)
i) (3, 7), (-2, 7)

II. Find the equation of the line on the points in I.

III. Under what circumstances are the lines $y = mx + b$ and $y = m'x + b'$ $(b \neq b')$

a) parallel?
b) perpendicular?

Solutions to Exercises 4-5a Pages 249-250

1. a) $\frac{2 - 1}{1 - 0} = 1$

b) $\frac{2^{.1} - 1}{.1 - 0} \approx \frac{.072}{.1} = .72$

c) $\frac{2^{-.1} - 1}{-.1 - 0} \approx \frac{.067}{.1} = .67$

$$2^{-.1} = 2^{-1} \cdot 2^{.9}$$
$$\approx \frac{1}{2}(1.866) = .933$$

d) $\frac{2^{-.01} - 1}{-.01 - 0} \approx \frac{.0068}{.01} = .68$

$$2^{-.01} = 2^{-1} \cdot 2^{.95} \cdot 2^{.04}$$
$$\approx \frac{1}{2}(1.932)(1.02819)$$
$$\approx .9932$$

A comparison reveals that both (a) and (b) gave larger results for $k \approx .69314$ while (c) and (d) gave smaller results.

Average of values for k in (b) and (c) $\frac{.72 + .67}{2} = \frac{1.39}{2}$ $= .695$ which is very close to $k \approx .693$.

2. Slope from graph $\approx .7$ $k \approx .693 \ldots$

3. $f : x \to 4^x = (2^2)^x = 2^{2x}$. If we number the scale for $f : x \to 2^x$ from 0 to 1.5, halving the existing scale on page 12 (T.C.), we get the graph of the function $x \to 4^x$. (See page 12 of T.C.)

   Slope for $x \to 4^x$ at $(0, 1)$ is $2k$ as the abscissas are divided by 2 while the ordinates are held fixed.

Solutions to Exercises 4-5b. Pages 252-253

1. $x \to 2^x$ has the slope $k \cdot 2^h$ at $(h, 2^h)$, $k \approx .69$.
   At $(1, 2)$ the slope is $k \cdot 2^1 = 2k \approx 2(.69) = 1.38 \approx 1.4$.
   You should get something close to 1.4 from the graph.
   In using the graph, use an interval of 1 or 2 for the abscissa to ease your arithmetic.

2.

Abscissa of Point	Slope $k \cdot 2^h$ for $x \to 2^x$ at $(h, 2^h)$	Slope $= 2x$ for $x \to x^2$
0	$k \approx .69$	0
1	$2k \approx 1.38$	2
2	$4k \approx 2.76$	4
-1	$\frac{1}{2}k \approx .35$	-2
-2	$\frac{1}{4}k \approx .17$	-4

3. Check graphically: Use the graph on page 12 of T.C.

4. As the slope of $x \to 2^x$ at $(h, 2^h)$ is approximately $.69 \cdot 2^h$, the slope is positive for real h so that the

rate of change is positive. Hence, the rate of change of $x \to 2^x$ is positive for all real x and is never 0 or negative. The slope function for $x \to x^2$ is $2x$. Hence the rate of change for $x \to x^2$ is positive, zero, or negative respectively.

The slope of $f : x \to a^x$ at (h, a^h) may be found as follows: the slope of the line containing (h, a^h), and $(h + u, a^{h+u})$ is

$$\frac{a^{h+u} - a^h}{(h + u) - h} = \frac{a^h(a^u - 1)}{u} = \frac{a^h(2^{\alpha u} - 1)}{u} = \alpha a^h\left(\frac{2^u - 1}{\alpha u}\right).$$

Hence, as u (or αu) approaches 0, the slope of the secant line approaches $\alpha a^h \cdot k$ which is the slope of the limiting secant or the tangent. It will be convenient to write the approximation $.69\alpha a^h$ where $a = 2^\alpha$, as the slope at (h, a^h).

Solutions to Exercises 4-5c — Pages 255-256

1. We have derived the slope at (h, a^h) for $f : x \to a^x$ as αka^h above. Hence, the equation of the tangent line through (h, a^h) is $y - a^h = \alpha ka^h(x - h)$ where $a = 2^\alpha$ and $k \approx .69$.
2. At $(1, 8)$ the slope of $f : x \to 8^x$ is found as follows. We have the general formula, slope $= k\alpha a^h \approx .69\alpha a^h$. For this problem, the slope $= k\alpha 8^1 = 8\alpha k$ where $2^\alpha = 8$. Hence $\alpha = 3$ and the slope at $(1, 8)$ is $8 \cdot 3 \cdot k = 24k$ $\approx 24(.69) \approx 16.5$ or 17 roughly.

At $(2/3, 4)$ we get for $\alpha k a^h$, $3 \cdot k \cdot 8^{2/3} = 3 \cdot k \cdot 4$ or approximately 8.3.

3. $f : x \to (\sqrt{2})^x = 2^{.5x}$. Hence $\alpha = .5$. The slope at $(2, 2)$ is $.5 \cdot k \cdot \sqrt{2}^{\,2} = .5k \cdot 2 = k$. Hence, the equation of the tangent at $(2, 2)$ is $y - 2 = k(x - 2)$ where $k \approx .69$.

4. $f : x \to 4^x = 2^{2x}$ so that $\alpha = 2$. At $x = 2$, the slope is $2 \cdot k \cdot 4^2 = 32k \approx 32(.69) \approx 22$.

) The graph on page 12 does not extend to $x = 2$. We may, however, find the slope for $x \to 4^x$ at $x = 1$ which is $2 \cdot k \cdot 2^2 = 8k$ and then multiply the answer by 4. If we do this we get $4 \cdot 5.5 = 22$ approximately. As the graph is very steep here it is acceptable to have an answer anywhere from 20 to 24.

5. We are given the slope at (h, a^h) for $x \to a^x$ as $2a^h$. Hence, $2a^h = \alpha \cdot k \cdot a^h$, so that $2 = \alpha k$ or

$$\alpha = \frac{2}{k} \approx \frac{2}{.69} \approx 2.9 \text{ . As } a = 2^{\alpha}, \quad a = 2^{2/k} \approx 2^{2.9}$$

$$= 2^2 \cdot 2^{.9} \approx 4(1.866) \approx 7.5 \text{ . Hence, } a \approx 7.5 .$$

6. The slope of $RS = \dfrac{a^{x+s} - a^x}{(x + s) - x} = \dfrac{a^x(a^s - 1)}{s}$.

The slope of $PQ = \dfrac{a^s - 1}{s - 0} = \dfrac{a^s - 1}{s}$.

The value of $f : x \to a^x$ at $R(x, a^x)$ is a^x. Hence the slope of RS = (value of f at R) $\cdot$ (slope of PQ)

or $\dfrac{a^x(a^s - 1)}{s} = a^x \cdot \dfrac{a^s - 1}{s}$.

7. The slope of the line containing $(0, 1)$ and $(0.01, 4^{0.01})$ is $\frac{4^{0.01} - 1}{.01 - 1} = \frac{2^{0.02} - 1}{.01} \approx \frac{1.01396 - 1}{.01} = \frac{.01396}{.01} = 1.396$.

At $(0, 1)$ the slope of the graph for $f : x \to 4^x$ is $2k$ which is approximately $2(.693) = 1.386$. Hence, the error is about .01 in 1.39 which is less than 1 percent.

8. It is possible to do this problem without the step by step graphing. Suppose the graph were made perfectly and that the sequence of points obtained is $(0, y_1)$, $(.1, y_2)$, $(.2, y_3)$, ... , $(1.0, 0, y_{11})$. Then

$y_1 = 1$

$y_2 = y_1 + 2(y_1)(.1) = 1 + .2 = 1.2$ (the .1 is the abscissa difference.)

$y_3 = y_2 + 2(y_2)(.1) = 1.2y_2 = (1.2)^2$

$y_4 = y_3 + 2(y_3)(.1) = 1.2y_3 = (1.2)^3$

$y_{11} = (1.2)^{10}$

The following table gives the coordinates:

n	x_n	$y_n = (1.2)^{n-1}$
1	0	1
2	.1	1.2
3	.2	1.44
4	.3	1.73
5	.4	2.07
6	.5	2.49
7	.6	2.98
8	.7	3.58
9	.8	4.29
10	.9	5.16
11	1.0	6.17
	-.1	.83
	-.2	.69

Plot these points. Then the line containing consecutive points is drawn for every pair. The set of lines then approximately encloses the graph for $f : x \rightarrow e^{2x}$ which will be discussed in the next section.

From the graph so obtained on page 21, for $x = 1$, $y \approx 6.2$, approximating (not too well) $e^2 \approx (2.72)^2 \approx 7.5$.

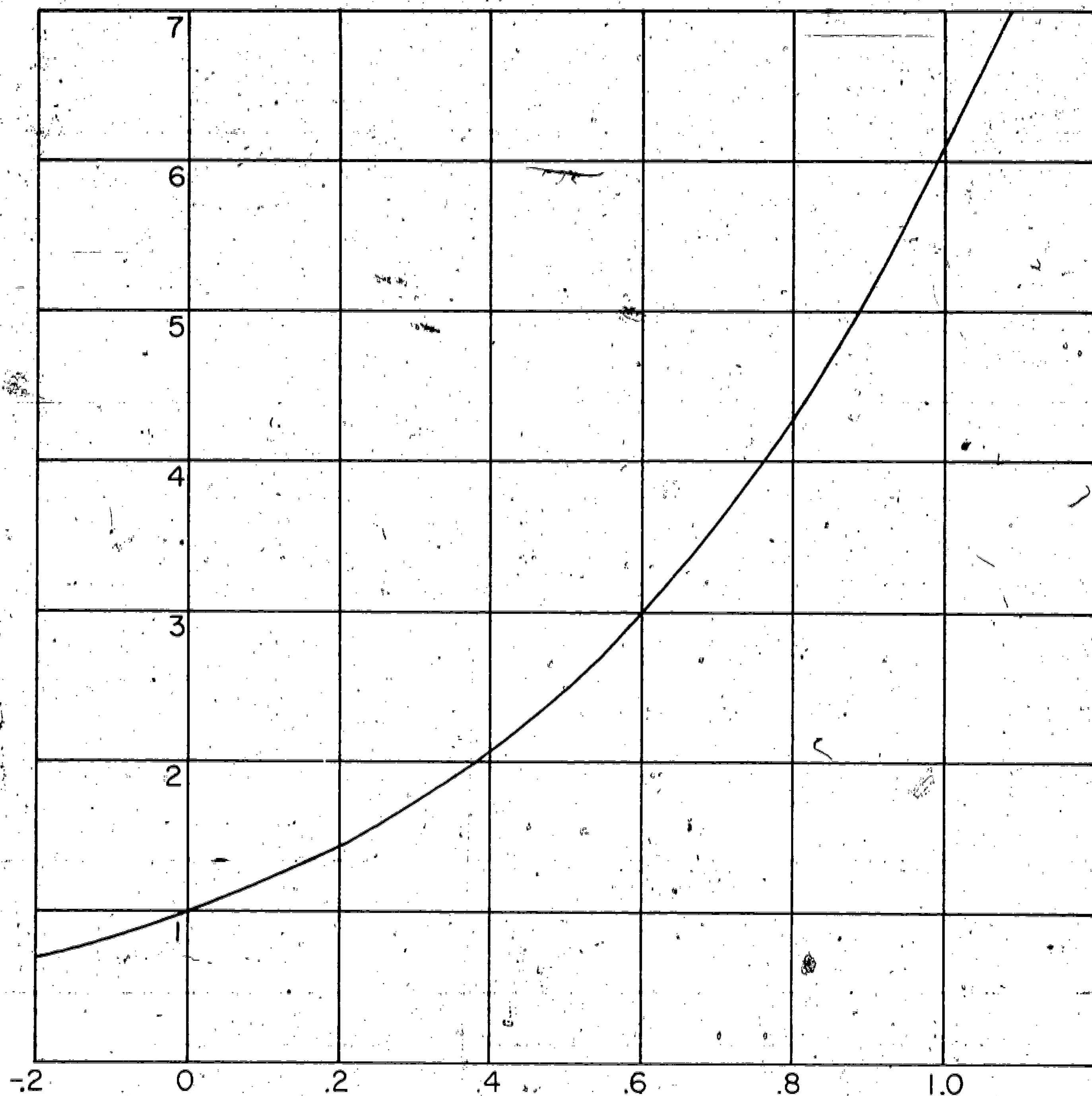
In Exercise 5, when $x = 1$, the corresponding point is at $(1, 7.5)$. Hence, $y \approx 7.5$ (which is approximately e^2).

9. This problem is like Exercise 8, but easier. Using a similar analysis we get

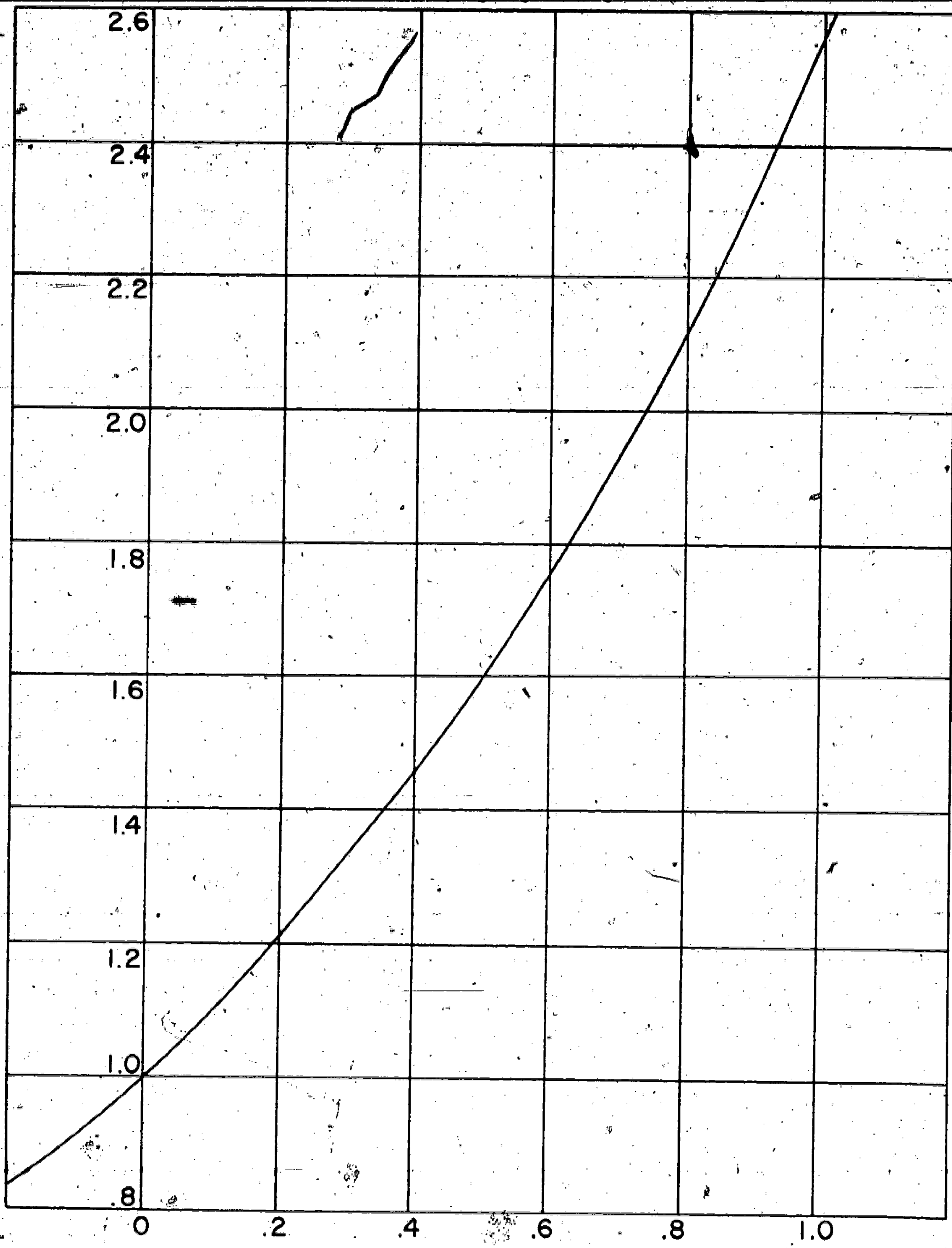
$$y_1 = 1$$
$$y_2 = y_1 + y_1(.1) = 1.1y_1 = 1.1$$
$$y_3 = y_2 + y_2(.1) = 1.1y_2 = (1.1)^2$$

$$y_{11} = (1.1)^{10}$$

Ex 4-5-C #8



Ex. 4-5-C #9



n	x_n	$y_n = (1.1)^{n-1}$
1	0	1
2	.1	1.1
3	.2	1.21
4	.3	1.33
5	.4	1.46
6	.5	1.61
7	.6	1.77
8	.7	1.95
9	.8	2.14
10	.9	2.36
11	1.0	2.59
	-.1	.91
	-.2	.83

Plot these points. Draw the lines containing pairs of consecutive points as in Exercise 8. These lines envelope the graph of $f : x \to e^x$ which will be discussed in the next session.

From the graph obtained, when $x = 1$, $y \approx 2.6$. This is a fair approximation to $e \approx 2.7$.

Solutions to Exercises 4-6 Pages 260-261

1. $(1 + 1/3)^3 = (4/3)^3 = 64/27 \approx 2.37$

   $(1 + 1/4)^4 = (5/4)^4 = 625/256 \approx 2.44$

   $(1 + 1/5)^5 = (6/5)^5 = (1.2)^5 \approx 2.49$

   $(1 + 1/6)^6 \approx (1.167) \approx 2.52$, off by about 0.2 from e.

2. For $n = 2$: $1 + 1 + 1/2 = 2.5$

   $n = 3$: $1 + 1 + 1/2 + 1/6 \approx 2.667$

   $n = 4$: $1 + 1 + 1/2 + 1/6 + 1/24 \approx 2.667 + .042 = 2.709$

   $n = 5$: $1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 \approx 2.709 + .008$

   $= 2.717$

As $e = 2.718$ the error $= 0.001$, approximately.

The work for $n = 5$ could have been arranged in the following manner, where the 3rd line is obtained from the 2nd line by dividing the 2nd line by 2. In general, the $(n + 1)$th line is obtained from the nth line by dividing the nth line by n. Thus

$$\begin{array}{r} 1 \\ 1 \\ .5 \\ .1667 \\ .0417 \\ .0082 \\ \hline 2.7166 \approx 2.717 \end{array}$$

3. a) $e^{.90} \approx 2.460$

$e^{.88} \approx 2.340 + d$

$e^{.85} \approx 2.340$

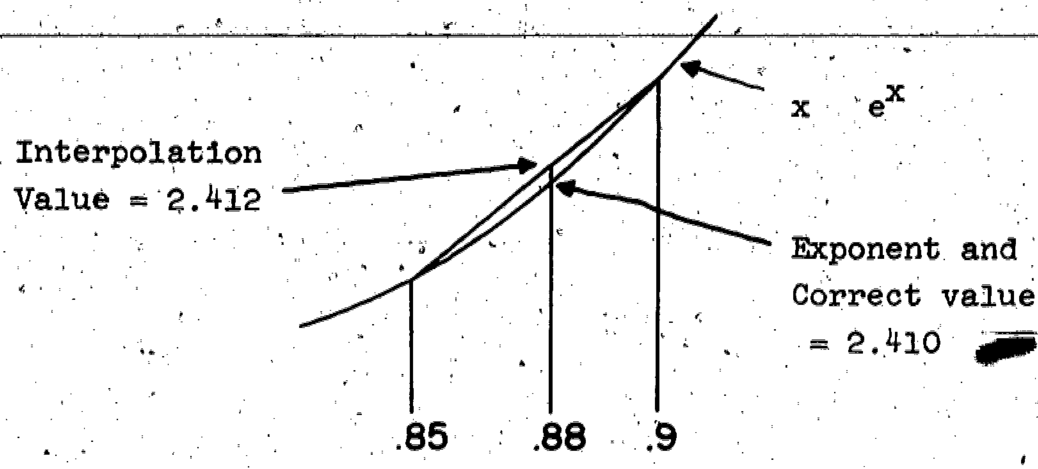
$3/5 = d/120$ (d is in thousandths)

$d = 72$

Hence, $e^{.88} \approx 2.340 + .072 = 2.412$

b) $e^{.88} = e^{.85} \cdot e^{.03} \approx (2.340)(1.030) \approx 2.410$

This value for $e^{.88} \approx 2.410$ is the better value as it was obtained by using the identity $e^a e^b = e^{a+b}$. In linear approximation it is assumed that the graph is approximated by a line segment in the interval under consideration. As $x \to e^x$ is a rapidly increasing function, it is concave upwards so that linear interpolation would yield a larger value. Thus:



(Use tables on page 262)

4. a) The slope at $(1, e)$ of $x \to e^x$ is $e \approx 2.72$.

The slope at $(3, e^3)$ of $x \to e^x$ is $e^3 \approx 20$.

b) The slope of the line containing $(1, e)$ and $(3, e^3)$

is $\dfrac{e^3 - e}{3 - 1} \approx \dfrac{20.086 - 2.718}{2} = \dfrac{17.368}{2} = 8.684$.

c) In other words, as the slope at x of $x \to e^x$ is e^x, we seek a solution to $e^x = 8.684$.

Using linear interpolation, we get

$$\begin{array}{ll} 12.182 & 2.50 \\ 8.684 & 2.00 + d \\ 7.389 & 2.00 \\ \hline \end{array}$$

$$\frac{1.295}{4.793} = \frac{d}{50} \quad \text{(d is in hundredths)}$$

$$d = \frac{64.75}{4.793} \approx 13$$

Hence, $x \approx 2.13$.

Solutions to Exercises 4-7a — Pages 268-269

1. We use the formula $W(x) = W(0)2^{-x/T}$. Then $\frac{W(x)}{W(0)} = 2^{-x/T}$ so that $\frac{W(7.7)}{W(0)} = 2^{-\frac{7.7}{3.85}} = 2^{-2} = \frac{1}{4}$ so that after 7.7 days we would expect 1/4 of the sample to remain. Note that we could have observed that after every 3.85 days we have 1/2 left. Hence, after 2(3.85) days we would expect (1/2)(1/2) = 1/4 left.

   Similarly $\frac{W(30.8)}{W(0)} = 2^{-\frac{30.8}{3.85}} = 2^{-8} = \frac{1}{256}$

2. In this problem we seek $\frac{W(x)}{W(0)} = 2^{-x/T}$ when $T = 26.8$ and $x = 13.4$ and 80.4 or $x/T = \frac{13.4}{26.8} = \frac{1}{2}$ and $x/T = \frac{80.4}{26.8} = 3$. Therefore $\frac{W(13.4)}{W(0)} = 2^{-1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx .707$

   $\frac{W(80.4)}{W(0)} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} = .125$.

3. We know $W(x) = W(0)2^{-x/T}$ where $\frac{W(12.2)}{W(0)} = \frac{1}{16}$, and we seek T. Hence $\frac{1}{16} = 2^{-\frac{12.2}{T}}$ or $2^{-4} = 2^{-12.2/T}$.

   Hence, $-4 = \frac{-12.2}{T}$ and $T = \frac{12.2}{4} = 3.05$ minutes.

4. Once again we use $W(x) = W(0)2^{-x/T}$. We are given that when $x = 1$, $\frac{W(1)}{W(0)} = \frac{49}{50}$. Therefore, $\frac{49}{50} = 2^{-1/T}$, or $T = \frac{100}{3} \approx 33$ years. Hence $W(t) = 2 \cdot 2^{-t/33}$ milligrams after t years.

5. $\frac{W(x)}{W(0)} = 2^{-x/T}$

$$\frac{W(3.36 \cdot 10^4)}{W(0)} = \frac{3}{4} = 2^{\frac{-3.36\cdot 10^4}{T}}$$

Therefore, $\frac{4}{3} \approx 1.333 \approx 2^{0.414} = 2^{\frac{3.36\cdot 10^4}{T}}$.

Hence $.414 = \frac{3.36 \cdot 10^4}{T}$

and $T = \frac{3.36}{.414} \cdot 10^4 \approx 8.15 \cdot 10^4$.

6. $\frac{W(x)}{W(0)} = 2^{-x/T}$

$$\frac{W(3,000)}{W(0)} = \frac{.277m}{m} = .277 \approx 2^{\frac{-3,000}{T}}$$

Hence, $2^{\frac{3,000}{T}} \approx \frac{1}{.277} \approx 3.614 = 2 \cdot 1.807$

$$\approx 2^1 \cdot 2^{.85} = 2^{1.85},$$

and $\frac{3,000}{T} \approx 1.85$, $T = \frac{3,000}{1.85} \approx 1,620$ years.

Hence $\frac{W(x)}{W(0)} = 2^{-x/1,620}$; $W(0) = 2$ milligrams

$$W(810) = 2 \cdot 2^{-810/1,620} = 2 \cdot 2^{-.5} = 2^{.5} = \sqrt{2}$$

≈ 1.4 milligrams.

An alternate solution is the following. We have

$.277 = 2^{-3,000/T}$

200

We seek $W(810) = 2 \cdot 2^{\frac{-810}{T}}$

$$= 2 \cdot \left(2^{\frac{-3,000}{T}}\right)^{\frac{810}{3,000}}$$

$$= 2 \cdot .277^{\frac{81}{300}} \qquad \frac{81}{300} = \frac{27}{100} = .27$$

$$\approx 2 \cdot .277^{.27} \qquad .277 = \frac{1}{3.614}$$

$$= 2 \cdot \frac{1}{3.614^{.27}} = 2 \cdot \frac{1}{(2 \cdot 1.807)^{.27}}$$

$$= 2 \cdot \frac{1}{(2 \cdot 2^{.85})^{.27}} \qquad \frac{2}{(2^{1.85})^{.27}} = \frac{2}{2^{.5}}$$

$$= 2^{.5} \approx 1.4 \text{ approximately.}$$

Hence, $W(810) = 1.4$ milligrams approx.

Solutions to Exercises 4-7b Page 271

1. $A = P\left(1 + \frac{r}{100n}\right)^{nt}$. If $r = 3$, $n = 2$, $t = 18$,

$A = 1000(1 + .015)^{2 \cdot 18} = 1000(1.015)^{36}$

$\log A = 3 + 36(.00647) = 3 + .23292$

$A = \$1,709.70$ to nearest 10¢.

$A = \$1,710$ to nearest \$1.

2. $A = 1,000(1 + .0075)^{4 \cdot 18} = 1,000(1.0075)^{72}$

$\log A = 3 + 72(.00325) = 3 + .23400$

$A = 1,714.00$ to nearest 10¢.

$A = \$1,714$ to nearest \$1.

3. $A = Pe^{rt}$

$A = 1{,}000e^{(.03)18} = 1{,}000e^{.54} = 1{,}000e^{.5}e^{.04}$

$A \approx 1{,}000(1.649)(1.041) \approx 1{,}717$

$A \approx \$1{,}717$ If 5 place tables are used.

$A \approx \$1{,}716$ correct to nearest \$1.

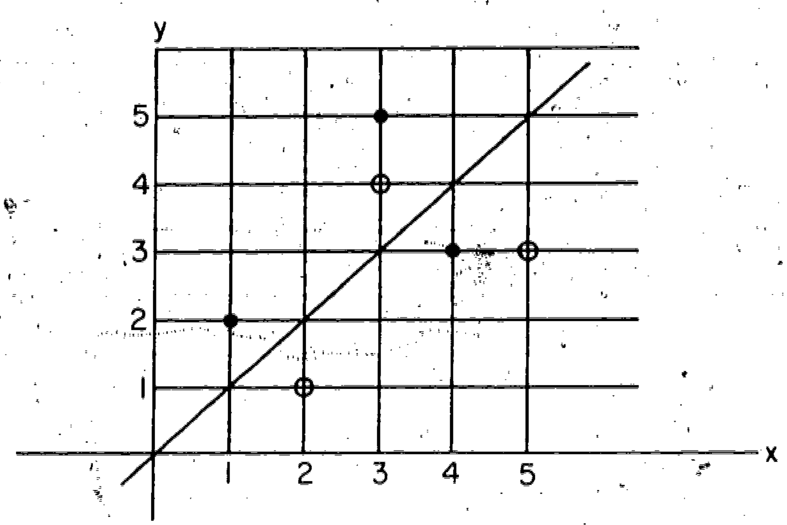
<u>Section 4-8.</u> It is frequently helpful in conveying the idea of an inverse to consider functions with a finite domain. Thus, let $f : x \to y$ be described by the table

x	1	3	4
y	2	5	3

Domain of $f = \{1, 3, 4\}$

Range of $f = \{2, 5, 3\}$

which we may represent on a graph as dots.



f^{-1} undoes what f does. Thus, if f sends 1 into 2, f^{-1} sends 2 right back to 1. Hence, the domain of f^{-1} is the range of f, $\{2, 5, 3\}$, and the range of f^{-1} is the domain of f. We may write the table for f^{-1} as

x	2	5	3
y	1	3	4

pictured on the above graph as circles. Note the symmetry of f and f^{-1} with respect to the graph of $y = x$.

When the expression on the right side of the arrow is simple, f^{-1} may be easily obtained from f. Thus, if $f : x \rightarrow 3x - 2$, then $f^{-1} : x \rightarrow \frac{x + 2}{3}$. f corresponds to the instructions "multiply by 3 and then subtract 2." To "undo" this and obtain f^{-1} we add 2 then divide by 3. This is all well and good for simple functions. However, this approach no longer works if $f : x \rightarrow \frac{x + 1}{x + 2}$.

If we write $f : x \rightarrow y$ where $y = \frac{x + 1}{x + 2}$, when we seek f^{-1} we want to find the value of x that is associated with a particular y. Hence, we solve $y = \frac{x + 1}{x + 2}$ for x, obtaining $x = \frac{2y - 1}{-y + 1}$. We then usually write $f^{-1} : x \rightarrow \frac{2x - 1}{-x + 1}$ using "x" in place of "y".

We could check this result quickly by taking a specific value for x, say $x = 0$, and seeing whether f^{-1} undoes what f does. Thus, $f : 0 \to \frac{0 + 1}{0 + 2} = \frac{1}{2}$ or $f(0) = 1/2$. f^{-1} should send $1/2$ back to 0. Let us see if it does. $f^{-1} : 1/2 \to \frac{2(1/2) - 1}{-(1/2) + 1} = \frac{1 - 1}{1/2} = 0$ and it does. In general we have $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(\frac{x + 1}{x + 2})$

$$= \frac{2\frac{x + 1}{x + 2} - 1}{-\frac{x + 1}{x + 2} + 1} = \frac{2(x + 1) - (x + 2)}{-(x + 1) + (x + 2)}$$

$$= \frac{2x + 2 - x - 2}{-x - 1 + x + 2} = \frac{x}{1}$$

$$= x .$$

Moreover, $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\frac{2x - 1}{-x + 1})$

or $$(f \circ f^{-1})(x) = \frac{\frac{2x - 1}{-x + 1} + 1}{\frac{2x - 1}{-x + 1} + 2} = \frac{2x - 1 + (-x + 1)}{2x - 1 + 2(-x + 1)}$$

$$\frac{x}{1} = x .$$

In general, if $f : x \to y = f(x)$, then solve the equation $y = f(x)$ for x in terms of y. This enables us to readily associate with a given y its x-partner and thus reveals the inverse, f^{-1}. Of course, if f does not have an inverse, the expression obtained for x in terms of y will reveal this.

The composition of two functions may be considered as following in succession two sets of instructions. Thus, if

$$f : x \to 3x \quad \text{and} \quad g : x \to x - 2$$

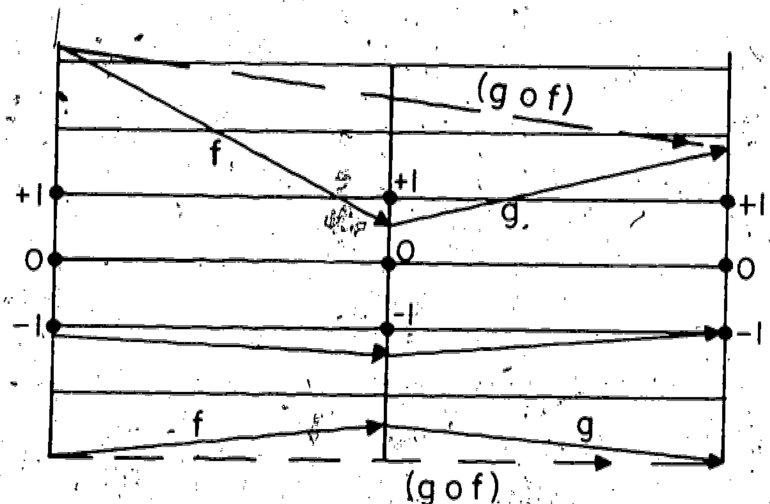
then $(g \circ f)$ instructs us to multiply x by 3 and then subtract 2. We could then write $(g \circ f) : x \to 3x - 2$. In more complex cases it is convenient to write $(g \circ f)(x) = g(f(x))$. In this illustration $g(f(x)) = g(3x) = 3x - 2$.

In the composition of functions it is most essential to observe the correct order. Thus, we found in our illustration that $(g \circ f): x \rightarrow 3x + 2$ However,

$$(f \circ g)(x) = f(g(x)) = 3(x - 2) = 3x - 6$$

so that $(f \circ g): x \rightarrow 3x - 6$. Hence, in general, $f \circ g \neq g \circ f$. One might ask the question, when does $f \circ g = g \circ f$ in the simple case where $f(x)$ and $g(x)$ are linear? You will find in this case that the line-graphs of f and g have slopes both 1, or they intersect on the line $y = x$.

We may visualize the composition of two functions, f and g as follows:



where the vertical lines represent number-lines, and the dashed arrows the composite function $(g \circ f)$.

Ref.: Appendices CEEB Commission Report, 1959, Pages 8 27

Solutions to Exercises 4-8

1. a) $x \to x + 7$;

b) $x \to \frac{x-9}{5}$;

c) $x \to \frac{1}{x}$;

d) $x \to \frac{3}{x-8}$.

One may do (d) as follows:

Set $y = \frac{3}{x} + 8$ and solve for x in terms of y, obtaining $x = \frac{3}{y-8}$. $y \to \frac{3}{y-8}$ is equivalent to $x \to \frac{3}{x-8}$.

Alternatively we may write successively

$f : x \to \frac{3}{x} + 8$;

$f : \frac{1}{x} \to 3x + 8$ (replacing x by $\frac{1}{x}$)

$f : \frac{3}{x} \to x + 8$ (replacing x by $\frac{x}{3}$)

$f : \frac{3}{x-8} \to x$ (replacing x by $x - 8$).

Hence, $f^{-1} : x \to \frac{3}{x-8}$.

2. a) $x = y + 7$ compared with $x \to x + 7$;

b) $x = \frac{y-9}{5}$ compared with $x \to \frac{x-9}{5}$;

c) $x = \frac{1}{y}$ compared with $x \to \frac{1}{x}$

d) $x = \frac{3}{y-8}$ compared with $x \to \frac{3}{x-8}$.

3. Let the number be x; then the various instructions given can be represented by the functions f_1 to f_7, as follows:

$$f_1 : x \to 5x$$
$$f_2 : x \to x + 6$$
$$f_3 : x \to 4x$$
$$f_4 : x \to x + 9$$
$$f_5 : x \to 5x$$
$$f_6 : x \to x - 165$$
$$f_7 : x \to \frac{x}{100}$$

Then

$$f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 : x \to \frac{5(4(5x + 6) + 9) - 165}{100} = x$$

and $f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_4^{-1} \circ f_6^{-1} \circ f_7^{-1} : x \to x.$

4. Suppose the digits are x and y, and we pick x. Then, as in Exercise 3:

$$f_1 : x \to 5x$$
$$f_2 : x \to x + 7$$
$$f_3 : x \to 2x$$
$$f_4 : x \to x + y$$
$$f_5 : x \to x - 14$$

$f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 : x \to 2(5x + 7) + y - 14 = 10x + y$, a number with tens digit x and units digit y.

5. $f : x \to \dfrac{2x - 3}{x + 4}$ $\qquad g : x \to \dfrac{x + 7}{3x - 1}$

a) $f \circ g : x \to \dfrac{2\dfrac{x + 7}{3x - 1} - 3}{\dfrac{x + 7}{3x - 1} + 4} = \dfrac{2x + 14 - 9x + 3}{x + 7 + 12x - 4}$

$f \circ g : x \to \dfrac{-7x + 17}{13x + 3}$

f^{-1} is found as follows:

Set $y = \dfrac{2x - 3}{x + 4}$ and solve for x in terms of y

$x = \dfrac{4y + 3}{-y + 2}$ hence $f^{-1} : x \to \dfrac{4x + 3}{-x + 2}$.

Similarly, g^{-1} is found as follows:

$y = \dfrac{x + 7}{3x - 1}$, $x = \dfrac{y + 7}{3y - 1}$, and $g^{-1} : x \to \dfrac{x + 7}{3x - 1}$.

6. $f : x \to \frac{ax + 1}{2x - 1}$

If $f = f^{-1}$, then $f \circ f : x \to x$

$$\frac{a\left(\frac{ax+1}{2x-1}\right) + 1}{2\left(\frac{ax+1}{2x-1}\right) - 1} \qquad \frac{a^2x + a + 2x - 1}{2ax + 2 - 2x + 1} = x \text{ identically.}$$

In particular this holds when $x = o$, in which case

$a - 1 = o$ or $a = 1$

$f : x \to \frac{x + 1}{2x - 1}$.

Check: $$(f \circ f)(x) = \frac{\frac{x+1}{2x-1} + 1}{2\,\frac{x+1}{2x-1} - 1} = \frac{x + 1 + 2x - 1}{2x + 2 - (2x - 1)}$$

$$= \frac{3x}{3} = x$$

so that $(f \circ f) : x \to x$

7. a) Suppose that f were not $1 - 1$. Then there could be 2 distinct x's, $x_1 < x_2$ such that $f(x_1) = f(x_2)$, contradicting the hypothesis that f is strictly increasing. Hence f is $1 - 1$ and therefore has an inverse by Theorem 4-4.

b) f is strictly decreasing if, whenever $x_1 < x_2$ are in the domain of f, $f(x_1) > f(x_2)$

c) A decreasing function is $1 - 1$. (The argument is similar to that given in a.) Hence, it has an inverse.

8. In this case, f is 1 - 1 and has an inverse. Suppose f were not 1 - 1. Then there would be two x's, $x_1 \neq x_2$ in the domain of f for which $f(x_1) = f(x_2)$. But then we would have a line parallel to the x-axis crossing the graph of f at 2 distant points $(x_1, f(x_1))$ and $(x_2, f(x_1))$. This contradicts the hypothesis on f. Hence f is 1 - 1 and has an inverse.

9. The hardest part of this problem is to keep track of the various domains.

a) We are given:

(1) $(gf)(x) = x$ for each x in the domain of f;

(2) $(fh)(y) = y$ for each y in the domain of h.

If $(fh)(y) = f(h(y))$ exists, $h(y)$ must be in the domain of f. Hence (2) implies that, for each y in the domain of h, $h(y)$ is in the domain of f and can therefore replace x in (1). We then have, for each y in the domain of h:

$(gf)(h(y)) = h(y)$ by (1)

$(gf)(h(y)) = g(f(h(y))) = g((fh)(y)) = g(y)$ by (2), and therefore $g(y) = h(y)$, on the domain of h, by substitution.

b) Just interchange g and h in (a).

c) If f and h are inverses, we must have

(3) $(fh)(x) = x$ for each x in the domain of h, and

(4) $(hf)(x) = x$ for each x in the domain of f.

Also, if g and g are inverses, we have

(5) $(fg)(x) = x$ for each x in the domain of g, and

(6) $(gf)(x) = x$ for each x in the domain of f.

From (3) and (6), using (a), we have $g(x) = h(x)$ for each x in the domain of h, and from (4) and (5), using (b), $h(x) = g(x)$ for each x in the domain of g. Hence g and h agree on both domains and must be the same function.

Section 4-9. Logarithms

If f is defined by $f : x \to 2^x$ we have, in particular, $f : 3 \to 2^3$ or $f: 3 \to 8$. Thus, under f, 3 is associated with 8. Under f^{-1} then 8 is associated with 3, or $f^{-1} : 8 \to 3$. In general, $f^{-1} : 2^x \to x$. This form is neither convenient nor in conformity with our way of writing a function. We prefer to write $f^{-1} : x \to y$ and represent y in some manner in terms of x.

We get around this by using a new symbol, $\log_2$, for f^{-1} so that we may now write $f^{-1} : x \to \log_2 x$. Thus $\log_2 8 = 3$ is simply another way of writing $2^3 = 8$. Students usually find it easy to understand by thinking of a relation, existing among 2, 3, and 8, as being expressed by two equivalent forms, logarithmic and exponential. In general, the two equivalent forms are

(1) logarithmic - $\log_B N = E$

(2) exponential - $B^E = N$

both expressing the very same relation among B, E, and N. As an illustration, consider the problem of finding $\log_4 64$. If we set $\log_4 64 = x$ and use the equivalent exponential form, $4^x = 64$, we are on home ground and easily obtain $x = 3$.

Solutions to Exercises 4-9, Pages 292-293.

1. $32 = 2^5 = 4^x = 2^{2x}$, hence $2x = 5$ and $x = 2.5$
2. $a \cdot a^m = a^{1+m} = (a^2)^m = a^{2m}$ $\therefore$ $2m = 1 + m$ and $m = 1$
3. $2^x = 10$ $x = 3.32$ approx.
4. a) $\log_2 2^3 = \log_2 8 = 3$

   b) $\log_2 2.5 \approx 1.32$, hence $1 < \log_2 2.5 < 2$

   c) $\log_2 x < 0$ for $0 < x < 1$ checks with graph
5. $\log (x \cdot \frac{1}{x}) = \log 1 = 0$

   But $\log (x \cdot \frac{1}{x}) = \log x + \log (\frac{1}{x})$;

   therefore $\log x + \log (\frac{1}{x}) = 0$ or $\log (\frac{1}{x}) = -\log x$
6. $\text{Log} (\frac{x_1}{x_2}) = \log (x_1 \cdot \frac{1}{x_2}) = \log x_1 + \log (\frac{1}{x_2})$

   $= \log x_1 - \log x_2$ from (5), for $x_1 > 0, x_2 > 0$
7. Recall $f : x \rightarrow a^x$, hence $f(1) = a$, $f^{-1}(a) = 1$, or

   $\log_a a = 1$.
8. a) $10^y = 35$

   b) $2^x = 25$

   c $q \log_c d = b$ or $\log_c d = \frac{b}{q}$ [or $\log_c d^q = b$];

   hence $c^{\frac{b}{q}} = d$ [or possibly $c^b = d^q$]
9. $\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2$

   $\approx 1 - .3010 = .6990$

   $\log_{10}(\frac{1}{2}) = -\log_{10} 2 \quad - .3010$

$\log_{10}(\frac{25}{4}) = \log_{10}(\frac{100}{16}) = \log_{10} 100 - \log_{10} 2^4$

$= 2 - 4 \log_{10} 2 \approx 2 - 4(.3010) = 2 - 1.2040$

$= .7960$

$\log_{10}(\frac{128}{5}) = \log_{10}(\frac{256}{10}) = \log_{10} 256 - \log_{10} 10$

$= \log_{10} 2^8 - 1 \approx 8(.3010) - 1$

$= 2.4080 - 1 = 1.4080$

10. a) $\log_2 5.5 \approx 2.46$

b) $\log_2(\frac{3}{4}) = \log_2 3 - \log_2 4 = \log_2 3 - 2 = 1.585 - 2 =$

$-.415$

Solutions to Exercises 4-10 Pages 299-301.

1. a) If $2^p = 26$ then $\log_2 26 = p$

b) If $\log_2 x = 5$ then $x = 2^5 = 32$

c) $\log_3 3^{\frac{1}{4}} = \frac{1}{4}$

d) $\log_2 (8 \times 16) = \log_2 8 + \log_2 16 = 3 + 4 = 7$

2. a) $\log_{10} 1000 = 3$

b) $\log_{.01} .001 = x$, hence $.01^x = .001$ or $10^{-2x} = 10^{-3}$, or $-2x = -3$ and $x = 1.5$

c) $\log_3(\frac{1}{81}) = -\log_3 81 = -\log_3 3^4 = -4$

d) $\log_4 32 = \log_4 2^5 = 5 \log_4 2 = 5(.5) = 2.5$

3. a) $4^{\log_4 5} + 3^{\log_3 5} = 2^{\log_2 x}$

or $5 + 5 = x$ and $x = 10$

b) $\log_{10}(x^2 - 1) - 2 \log_{10}(x - 1) = \log 3,$

$$\log_{10}(x^2 - 1) + \log_{10}(x - 1)^{-2} = \log 3, \qquad x \neq 1$$

$$\log_{10} \frac{x^2 - 1}{(x-1)^2} = \log 3,$$

$$\log_{10} \frac{x + 1}{x - 1} = \log 3,$$

$$\frac{x + 1}{x - 1} = 3, \text{ and } x = 2$$

c) $7^{\log_x 5} = 5$, hence $x = 7$

4. $e^{(\ln 10)(\log_{10} e)} = (e^{\ln 10})^{\log_{10} e} = 10^{\log_{10} e} = e$

Hence $\ln(10^{\log_{10} e}) = \ln e$

or $(\log_{10} e)(\ln 10) = 1$

5. a) If $\log_c x = 0$, $x = 1$

b) If $\log_x x = 1$, $x > 0$

c) If $x^{\log_x c} = c$, $x > 0$, $x \neq 1$

d) If $\log_x 2^x = 2$,

$$x^2 = 2^x \text{ and } (x^2)^{\frac{1}{2x}} = (2^x)^{\frac{1}{2x}}$$

or $x^{\frac{1}{x}} = 2^{\frac{1}{2}}$

$\therefore$ $x = 2$, $x = 4$ See Misc. Ex. P. 310 #36

6. $\log_{10} 5 = \log_{10}(\frac{10}{2}) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2.$

Hence $(\log_{10} 2)(1 - \log_{10} 2) = .210$. Let $y = \log_{10} 2$.

Then $y(1 - y) = .210$

$y^2 - y + .210 = 0$

$(y - .7)(y - .3) = 0$

$y = .7$, $y = .3$

But $\log_{10} 2 < .5$ as $2 < 10^{.5} = \sqrt{10}$.

Hence $y = .3$

or $\log_{10} 2 = .3$ (approx.)

7. If $f : x \to 1^x$ $f(2) = f(3)$ and f has no inverse. The $\log_a$ is the inverse of the exponential, $x \to a^x$.

8. $\log_{10} x = x$ has no solution. An easy way to see this is to graph $y = x$, $y = \log_{10} x$. The graphs do not intersect. (See Figure 4-9a.) Also, if $\log_{10} x = x$

then $\frac{1}{x} \log_{10} x = 1$

or $\log_{10} x^{\frac{1}{x}} = 1$

or $x^{\frac{1}{x}} = 0$

and there is no real x satisfying this equation.

9. $(\ln x)^2 = \ln x^2$

or $(\ln x)^2 = 2 \ln(x)$.

If $\ln x = 2$, $x = e^2$

$\therefore$ Solution Set = $\{1, e^2\}$

10. $2N = N(1 + .04)^x$

$2 = (1.04)^x$

$x = \frac{\log 2}{\log 1.04} \approx 17.6$ yrs. or 18 yrs.

$2N = N(1.0075)^{4x}$

$4x = \frac{\log 2}{\log 1.0075} \approx \frac{.30103}{.00325}$ or $x \approx \frac{.301}{.013} = \frac{301}{13}$

$x \approx 23.2$ or $23\frac{1}{4}$ yrs.

11. $2N = N(1 + \frac{x}{100} \cdot \frac{1}{4})^{4(10)}$

$2 = (1 + .0025x)^{40}$

$\log_{10}(1 + .0025x) = \frac{\log_{10} 2}{40} \approx \frac{.30103}{40} \approx .007526$

$1 + .0025x \approx 1.0175$

$.0025x \approx .0175$

$x \approx 7$ percent

12. In particular, the chapter, "The Next Generators of Computers", makes a nice project for a student to report on.

Miscellaneous Exercises

1. a) $N = A\ e^n$

$2A = A\ e^n$

$e^n = 2 \approx e^{0.693}$

$n \approx 0.693\ (= k)$

or $n = \ln 2 = k.$

b) Ratio = e Hence percentage increase

$= 100\ (e-1) \approx 172$ percent

2. $f(0) = ca^0 = c = 2$

$f(1.5) = ca^{1.5} = 2a^{1.5} = 54$

Hence $a^{3/2} = 27$, $a = 9.$

3. $f(2) = a^2 = 0.25$ Hence $a = 0.5$

$f : x \longrightarrow (\frac{1}{2})^x = \frac{1}{2^x}$

$f(5) = \frac{1}{2^5} = \frac{1}{32}$

4. a) $e^{\ln x} = x$

b) $\ln(e^x) = x$

5. $s = \frac{a - ar^n}{1 - r}$, $r \neq 1$

$ar^n = a - s(1-r)$

$$n = \frac{\log_{10} [1 - \frac{s(1-r)}{a}]}{\log_{10} r}$$

6 a) $\ln ab = \ln a + \ln b = 3 + 2 = 5$

b) $\ln a \cdot \ln b = 3 \cdot 2 = 6$

c) $\ln \frac{a}{b} = \ln a - \ln b = 3-2 = 1$

d) $\frac{\ln a}{\ln b} = \frac{3}{2}$

e) $\ln a^2 = 2\ln a = 2 \cdot 3 = 6$

f) $(\ln a)^2 = 3^2 = 9$

7. $\frac{V_1}{V_2} = \left(\frac{p_1}{p_2}\right)^{1/n}$. Hence $V_1 = V_2 \left(\frac{p_1}{p_2}\right)^{1/n}$

8. a) $x = 7$

b) $x = 2$

c) $x = 4$

d) $x = 7$

9. $a^{0.3} = r$ $\qquad \log_r a^{0.3} = \log_r r$

$0.3 \log_r a = 1$

$\log_r a = \frac{10}{3}$

10. a) $\ln 3 - \ln 5 + 2(\ln 5 + \ln 2) - (\ln 3 + 2\ln 2)$

$= \ln 5$

Alternately, $\ln\left(\frac{3}{5} \cdot 100 \cdot \frac{1}{12}\right) = \ln\frac{300}{60} = \ln 5.$

b) $\ln\left(x^2 \cdot \frac{1}{y^{1/4}} \cdot y^{\frac{2}{3}} \cdot \frac{1}{x^{\frac{1}{2}}}\right) = \ln\left(x^{\frac{3}{2}}\, y^{\frac{5}{12}}\right)$

11. a) $128 = 8^{7/3}$

b) $(1000)^{\frac{1}{3}} = 10$

c) $(27)^{\frac{2}{3}} = 9$

d) $(8/27)^{\frac{1}{3}} = 2/3$

e) $(0.1)^2 = 0.01$

12. a) $\log_{125} 5 = 1/3$

b) $\log_{10} (0.01) = -2$

c) $\log_{27} 81 = 4/3$

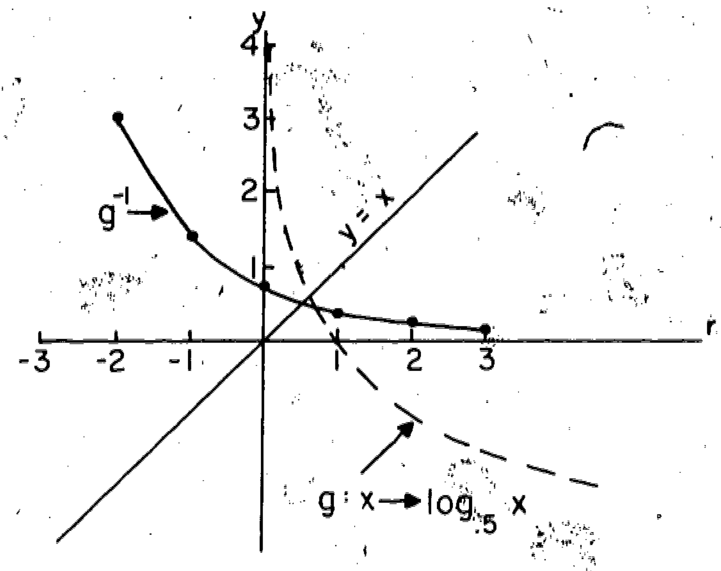
d) $\log_{0.04} (0.008) = 3/2$

e) $\log_{16} 2 = 1/4$.

13. $g : x \to \log_{1/2} x$ $\qquad g^{-1} : x \to (\frac{1}{2})^x$

$g : (\frac{1}{2})^x \to x$ $\qquad g^{-1} : x \to \frac{1}{2^x}$

r	2^r	$\frac{1}{2^r}$
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
0	1	1
-1	$\frac{1}{2}$	2
-2	$\frac{1}{4}$	4



14. The graphs of $x \to a^{-x}$ and $x \to a^x$ are symmetric with respect to the y - axis.

15. a) $x \to -2^x$ and $x \to 2^{-x}$ are symmetric with respect to the origin.

b) $x \to 2^{x}$ and $x \to 2^x$ are identical.

16. a) $(fg)(2) = f(g(2)) = f(3^2) = f(9) = 2^9 = 512$

b) $(gf)(2) = g(f(2)) = g(2^3) = g(8) = 3^8 = 6561$

17\. a) $f(x) + g(x) = 2 \cdot 2^x = 2^{x+1}$

b) $f(x) \cdot g(x) = (2^x)^2 - (2^{-x})^2 = 2^{2x} - 2^{-2x}$

c) $[f(x)]^2 - [g(x)]^2 = [f(x) + g(x)]\,[f(x) - g(x)]$

$= 2^{x+1}\,(2 \cdot 2^{-x}) = 4$

18\. Using $A_t = P\,e^{rt/100}$

$100 = P\,e^{5 \cdot 5/100} = P\,e^{\frac{1}{4}}$

$P = 100\,e^{-\frac{1}{4}} \approx 100(0.779) = 77.9$

$P = \$77.90 \approx \$78$

19\. $200 = 100\,(1 + x/100)^{10}$

$2 = (1 + x/100)^{10}$

$x = 100\,(2^{.1} - 1) \approx 100\,(1.072 - 1)$

≈ 7.2

Rate is about 7 percent.

20\. $2a = a\,\left(1 + \frac{x}{100 \times 12}\right)^{12 \times 10}$

$2 = \left(1 + \frac{x}{1200}\right)^{120} = \left[\left(1 + \frac{x}{1200}\right)^{\frac{1200}{x}}\right]^{\frac{x}{10}}$

$2 \approx e^{\frac{x}{10}} \approx e^{0.69}$

Hence $x \approx 6.9$ Rate 6.9 percent $\approx$ 7 percent

21\. $100\,e^{5(.05)} = 100\,e^{0.25} \approx 100\,(1.284) = 128.4$

Answer: \$128

22\. $2P = P\,e^{rt/100}$

$2 = e^{rt/100}$

$t = \frac{100}{r}\,\ln 2 \approx \frac{69.3}{r}$

a) 23.1 yr. $\approx$ 23 yr.

b) 11.55 yr. $\approx$ 12 yr.

c) $\frac{69.3}{n}$ yr.

23. Since

$\log_b x = (\log_a x)(\log_b a)$, $(\log_a b \cdot \log_b c)(\log_c d)$

$= (\log_a c)(\log_c d)$

$= \log_a d.$

24\. I. $\log_{10} e = \frac{1}{\log_e 10} = \frac{1}{\ln 10} = \frac{1}{2.3} \approx 0.43$

II. From Tables

$\log_{10} 2.713 \approx 0.4343$

III. $\log_{10} e = \frac{\log_2 e}{\log_2 10} = \frac{1.44}{3.32} \approx 0.43$

25\. $\ln\left(\frac{1-x}{1+x}\right) = 1 = \ln e,$

$\frac{1-x}{1+x} = e$ and

$x = -\left(\frac{e-1}{e+1}\right)$

26\. a) $\log_2 [(x-2)x] = \log_2 8$ and $x > 0.$

$\therefore x^2 - 2x - 8 = 0$

$x = 4$ or -2

Since $x > 0$, solution set is $\{4\}$

b) $\log_6 [(x+9)x] = \log_6 36$ and $x > 0.$

$x^2 + 9x - 36 = 0$

$x = 3$ or -12

Answer: $x = 3$

26. (c) $\log_2(6x + 5)x = \log_2 4 \qquad x > 0$

$$6x^2 + 5x - 4 = 0$$

Positive root is $x = 1/2$.

27. Let $2^x = y$

Then $2^2y^2 = 9y - 2$

$$4y^2 - 9y + 2 = 0$$

$$y = 1/4 \text{ or } y = 2$$

Hence

$$x = -2 \text{ or } x = 1$$

28. $\ln\left(\frac{x-4}{x+1}\right) = \ln 6$ means

$$\frac{x-4}{x+1} = 6$$

whose solution is $x = -2$

But for $x = -2$, $\ln(x-4)$ and $\ln(x+1)$ are undefined.

29. a) If $2^x = 2^{3x-1}$, $x = 3x - 1$ and $x = 1/2$

The intersection occurs at $(1/2, \sqrt{2})$.

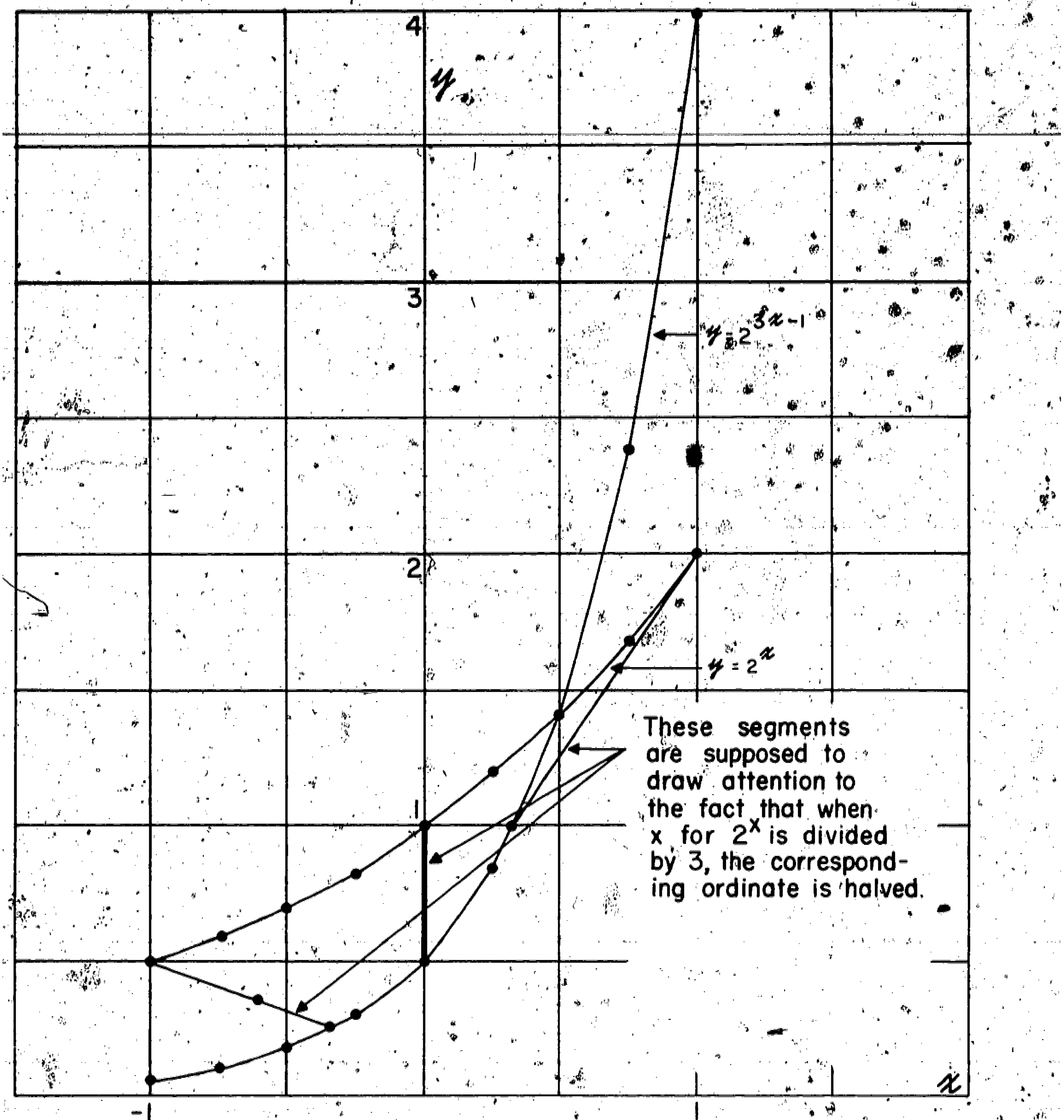
b) See graph

c) $2^{3x-1} = \frac{2^{3x}}{2}$

Hence, for $-1/3 \leq x \leq 1/3$,

the values of 2^{3x-1} are $1/2$ the values of

2^x on the interval $-1 \leq x \leq 1$.

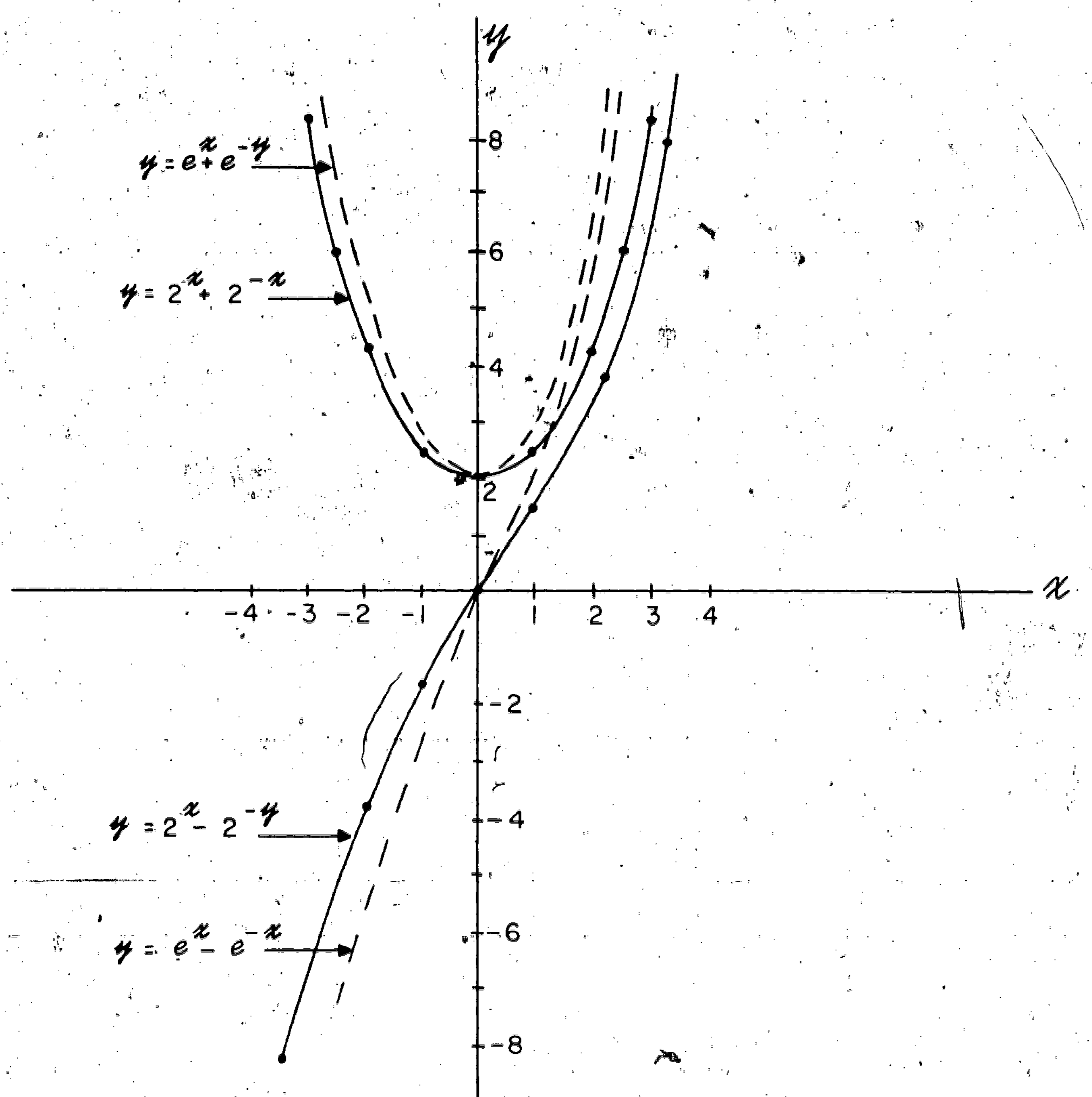

4
y
3
$y = 2^{3x-1}$
2
$y = 2^x$
These segments are supposed to draw attention to the fact that when x for 2^x is divided by 3, the corresponding ordinate is halved.
1
x
-1
1

30.	Range	one to one
a)	$\{x : x \geq 2\}$	No
b)	all reals	Yes
c)	$\{x : x \geq 2\}$	No
d)	all reals	Yes

To find the range in (a) solve $y = 2^x + 2^{-x}$ for 2^x, obtaining $2^x = \frac{y \pm \sqrt{y^2 - 4}}{2}$.

In any case $y > 0$. Since 2^x must be real, $y^2 \geq 4$. Hence $y \geq 2$.

A similar discussion may be made for (c).


y
x
$y = e^x + e^{-y}$
$y = 2^x + 2^{-x}$
$y = 2^x - 2^{-y}$
$y = e^x - e^{-x}$
8
6
4
2
-2
-4
-6
-8
-4
-3
-2
-1
1
2
3
4

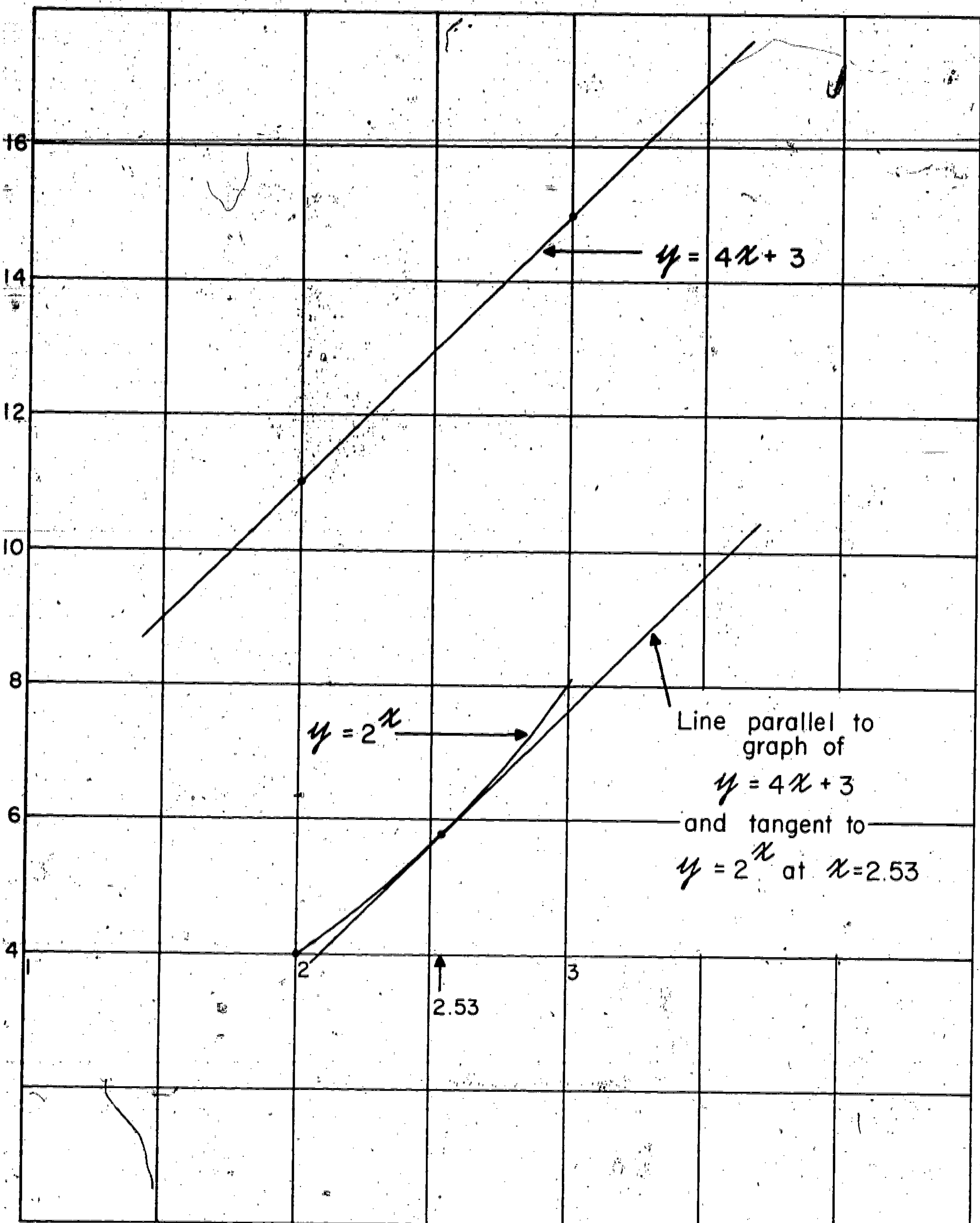
31. The step from line 3 to line 4 is fallacious since either log (a-b) or log (b-a) is meaningless because either $a - b < o$ or $b - a > o$.

32. The rate of change of 2^x is $k\,2^x$ at x.
The rate of change of $4x + 3$ is 4 at x.
If $k\,2^x = 4 = 2^2$

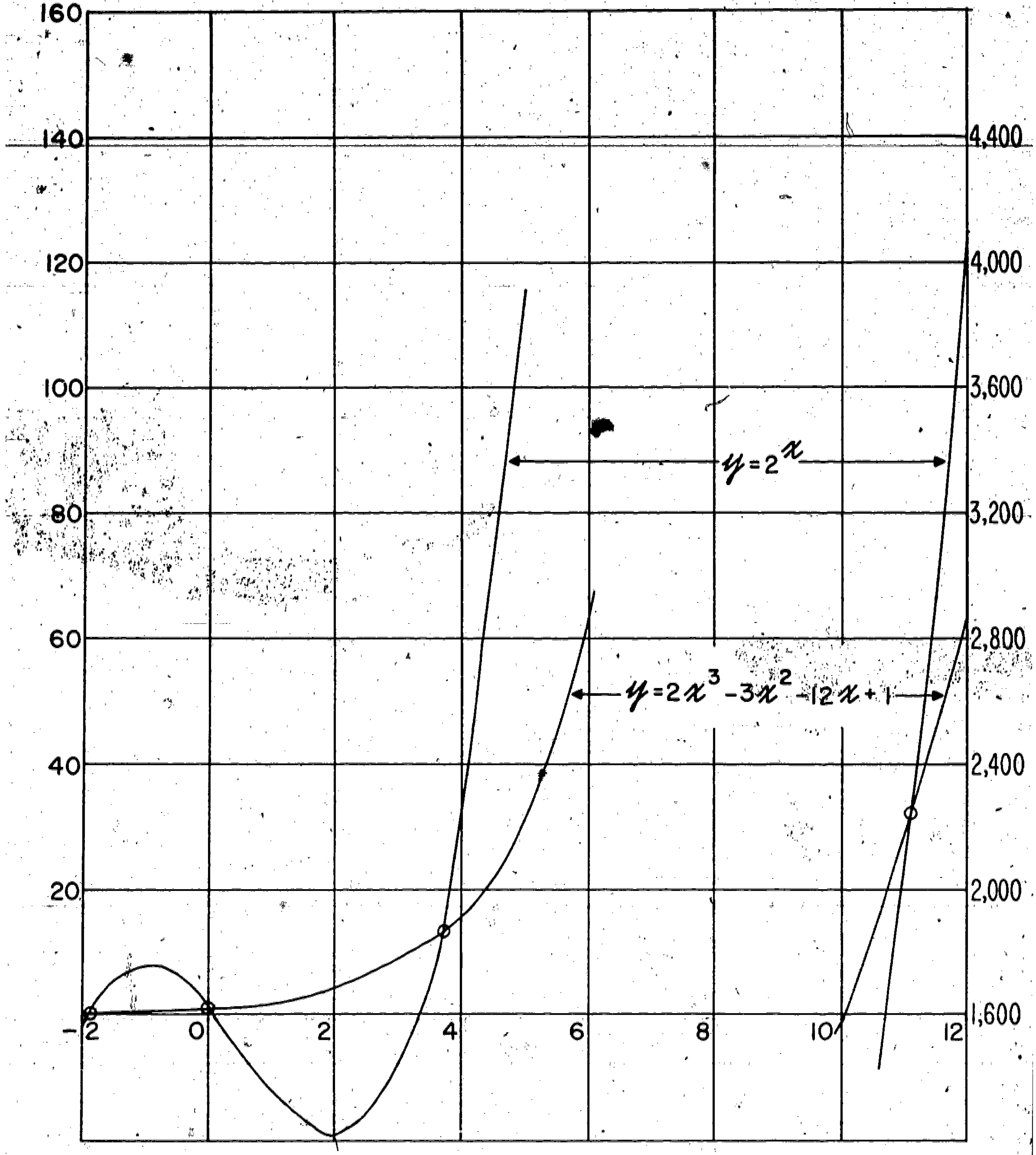
$$2^{x-2} = \frac{1}{k} \approx 1.443 \approx 2^{0.53}$$

Hence $x - 2 \approx 0.53$

$$x \approx 2.53$$



16
14
12
10
8
6
4
1
2
2.53
3
y = 4x + 3
y = 2^x
Line parallel to graph of
y = 4x + 3
and tangent to
y = 2^x at x=2.53


160
140
120
100
80
60
40
20
4,400
4,000
3,600
3,200
2,800
2,400
2,000
1,600
-2
0
2
4
6
8
10
12
$y = 2^x$
$y = 2x^3 - 3x^2 - 12x + 1$

33. If $(2^a)^b = (2)^{a^b}$,

$ab = a^b$

and $b = a^{b-1}$

If $b = 3$, $3 = a^2$ and $a = 3^{\frac{1}{2}}$

If $b = 4$, $4 = a^3$ and $a = 4^{\frac{1}{3}}$

34. If $\ln(x+y) = \ln x + \ln y$. $\ln(x + y) = \ln xy$

and $x + y = xy$

or $y = \frac{x}{x-1} = 1 + \frac{1}{x-1}$

If y is an integer, $\frac{1}{x - 1}$ must be an integer.

The only possibilities are $x - 1 = +1$ and $x - 1 = -1$, that is, $x = 2$ and $x = 0$. The last value must be rejected since $\ln 0$ does not exist. When $x = 2$, $y = 2$

Hence $(2, 2)$ is the only solution.

35. a) See graph.

b) Three

c) Three

d) Four, four

36. We take $b > a$.

An obvious solution is $a = 2$, $b = 4$ since

$$2^4 = 4^2 = 16.$$

It is not hard to show that there can be no integral solution of

$$a^b = b^a$$

unless both a and b are positive.

Taking logarithms to the base e

$$b \ln a = a \ln b$$

or

$$\frac{\ln a}{a} = \frac{\ln b}{b}$$

On the graph of $y = \frac{\ln x}{x}$ we are seeking two points with the same y and with x an integer in each case.

The graph of $y = \frac{\ln x}{x}$ rises from 0 at $x = 1$ to a maximum for x somewhat less than 3 (actually at $x = e$) and then decreases. If there are to be two points on the same level one must be to the left of $x = 3$. Of the possible integral values $x = 1$ may be excluded immediately leaving only $x = 2$, for a possible a. We know of course that this works.

Indeed $\frac{\ln 2}{2} = \frac{\ln 4}{4}$

Illustrative Test Questions

1. If in a bacterial experiment there were 50,000 bacteria at 12 noon and 73,205 at 8 p.m. of the same day, how many bacteria were there at

    (a) 6 p.m. of the same day, and

    (b) 8 a.m. of the same day?

2. If the number of bacteria at time t is given by the equation $N = 500,000\ (8)^{0.5t}$, find

    (a) the number present when $t = 1\frac{1}{3}$,

    (b) the number present when $t = -\frac{2}{3}$,

    (c) the time at which there are 31,250 present.

3. Evaluate (a) $4^{4/3}(8^{-2/3})^{1/3}$ and (b) $(16^{1/3}\ 25^{-4/3})^{-3/4}$.

4. Arrange the following in order of magnitude, agreeing that $a^{b^c} = a^{(b^c)}$:

    (a) 2^{3^4} (b) 2^{4^3} (c) 3^{2^4} (d) 4^{2^3} (e) 4^{3^2}

5. Using Table 4-2, compute

    (a) $2^{1.85}$

    (b) $2^{-2.70}$

    (c) $(0.125)^{-5.55}$

6. Using Table 4-2, compute

    (a) $(1.11)^{2.4}$

    (b) $(0.87)^{3.5}$

    (c) $(0.66)^{-4.50}$

7. Given the function $f : x \rightarrow 3^x$ and the points $A(0,1)$, $B(1,3)$, $C(a, 3^a)$, and $D(a+1, 3^{a+1})$ on the graph of f. Find

   (a) the slope of AB

   (b) the slope of CD

   (c) the value of a for which the slope of CD is 9 times the slope of AB.

8. Given the function of $f : x \rightarrow 2^x$, find

   (a) the slope of f at $x = 0$

   (b) the slope of f at $x = 1$

   (c) the slope of the line through $(0,1)$ and $(1,2)$

   Use $k = 0.7$ as an approximation.

9. Using $k = 0.7$ as an approximation, compute the slope of the tangent to the graph of $f : x \rightarrow 3(2)^x$ at the points $(0,3)$ and $(4,48)$.

10. Using Table 4-6a, compute

    (a) $e^{0.32}$

    (b) the slope of the curve $x \rightarrow e^x$ at $(0.3, e^{0.3})$

    (c) the slope of the secant through $(0.3, e^{0.3})$ and $(0.32, e^{0.32})$.

11. (a) The half-life of a radioactive substance is 4 days. What fraction of a sample of this substance <u>has decomposed</u> after 8 days?

    (b) If $\frac{15}{16}$ of a radioactive substance <u>has decomposed</u> after 3 days, what is its half-life?

12. If \$200 amounts to \$326 in 7 years at 7 percent compounded monthly, what will \$200 amount to in 14 years at the same rate of interest?

13. Given $f : x \rightarrow 2x$ and $g : x \rightarrow 3x - 1$, find

(a) $(f \circ g)^{-1}$

(b) $f^{-1} \circ g^{-1}$

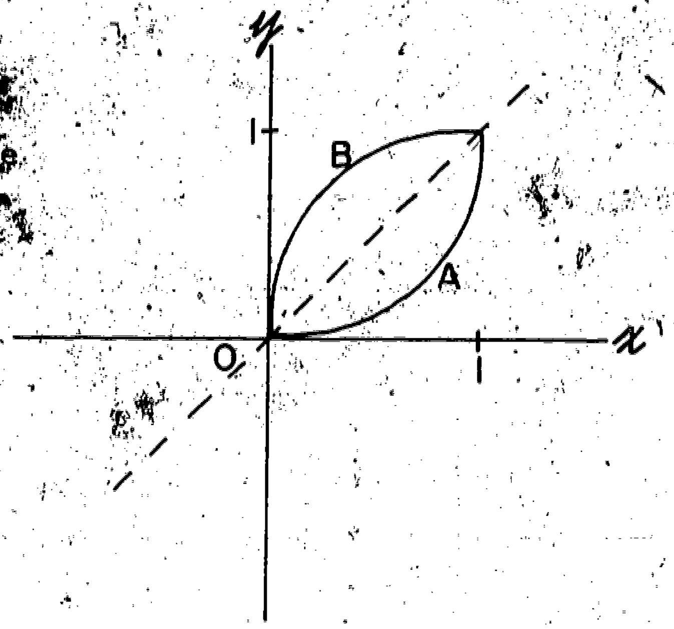
(c) $g^{-1} \circ f^{-1}$

14. In the figure at the right, arc A is the graph of the function

$f : x \rightarrow 1-\sqrt{1-x^2}$,

$0 \leq x \leq 1$.

If arc B is symmetric to A about the line $y=x$, what function is B the graph of?



15. Express each of the following in exponential form:

(a) $\log_2 16 = 4$,

(b) $\log_a x + \log_a y = z$.

16. Find x in each of the following:

(a) $\log_4 \sqrt{2}\,\sqrt{2} = x$,

(b) $\log_{2x} \sqrt{2} = \frac{1}{3}$,

(c) $\log_x (3x^2) = 5$.

17. Given $\ln a = 3$ and $\ln b = 5$, compute

(a) $\ln a^2 b$

(b) $\ln (\frac{a}{b^2})$

(c) $(\ln a)(\ln b)$

(d) $\ln a^{\ln b}$

(e) $\frac{\ln a}{\ln b}$

18. Given the functions $f : x \rightarrow 2^x$ and $g : x \rightarrow \log_4 x$, find

    (a) $(f \circ g)(3)$

    (b) $(g \circ f)(3)$

    (c) $(f^{-1} \circ g^{-1})(3)$

    (d) $(g^{-1} \circ f^{-1})(3)$

19. Solve for x :

    (a) $\log_6(x-4) + \log_6(x+3) = 1$

    (b) $\log_{15}(2x+3) + \log_{15}(x+1) = 1$

20. Solve for x :

    $$2^{2x+2} + 2^{x+2} = 3.$$

Test Questions, Chapter 4

1. $N = N_0 \, 2^{\alpha x}$

$73,205 = 50,000 \cdot 2^{8\alpha}$

$2^{8\alpha} = 1.4641$

$8\alpha \approx 0.55$

$\alpha \approx 0.069$

$N \approx 50,000 \cdot 2^{0.069x}$

(a) $50,000 \cdot 2^{0.069 \cdot 6} = 50,000 \cdot 2^{0.41} \approx 50,000\,(1.320)(1.00695)$

$66,500$

(b) $50,000 \cdot 2^{0.069 \cdot 20} = 50,000 \cdot 2^{1.38}$

$\approx 50,000(2)(1.275)(1.02101)$

$\approx 130,000$

2. $8^{0.5t} = (2^3)^{0.5t} = 2^{1.5t}$

(a) $500,000 \cdot 2^{1.5(1\frac{1}{3})} = 500,000 \cdot 2^2 = 2,000,000$

(b) $500,000 \cdot 2^{-1} = 250,000$, or $\frac{4}{3}$

(c) $31,250 = 500,000 \cdot 2^{1.5t}$

$2^{1.5t} = .0625 = 2^{-4}$

$1.5t = -4$

$t = -2\frac{2}{3}$

3. (a) $4^{\frac{4}{3}} (8^{-\frac{2}{3}})^{\frac{1}{3}} = 2^{\frac{8}{3}} \cdot (2^{-2})^{\frac{1}{3}} = 2^{\frac{8}{3}} \cdot 2^{-\frac{2}{3}} = 2^2 = 4.$

(b) $(16^{\frac{1}{3}} \, 25^{-\frac{4}{3}})^{-\frac{3}{4}} = 16^{-\frac{1}{4}} \, 25^1 = 2^{-1} \cdot 25 = 12\frac{1}{2}$

4. (a) $2^{3^4} = 2^{81}$ (b) $2^{4^3} = 2^{64}$ (c) $3^{2^4} = 3^{16} = (2^{1.58})^{16}$

$\approx 2^{25.3}$

(d) $4^{2^3} = 4^8 = 2^{16}$ (e) $4^{3^2} = 4^9 = 2^{18}$

The expressions given increase in the order (d), (e), (c), (b), (a).

5. (a) $2^{1.85} \approx 2(1.803) \approx 3.60$

(b) $2^{-2.70} = \frac{2^{.30}}{2^3} \approx \frac{1.231}{8} \approx 0.154$

(c) $(0.125)^{-5.55} = (1/8)^{-5.55} = (2^{-3})^{-5.55} = 2^{16.65}$

$\approx (65,536)(1.569)$

$\approx 103,000$

6. (a) $(1.11)^{2.4} = (2^{.15})^{2.4} = 2^{.36} \approx (1.275)(1.00695) \approx 1.28$

(b) $(0.87)^{3.5} = (\frac{1}{0.87})^{-3.5} \approx (1.15^{-})^{-3.5} \approx (2^{0.20})^{-3.5}$

$= 2^{-.7} = \frac{2^{.3}}{2} \approx \frac{1.231}{2} \approx 0.62$

(c) $(0.66)^{-4.5} \approx (1.5)^{4.5} \approx (2^{0.6})^{4.5} = 2^{2.7} \approx 6.5$

or $\approx (1.53)^{4.5} \approx (2^{0.61})^{4.5} = 2^{2.75}$

$\approx (1.682)(4) \approx 6.73$

7. (a) $\frac{3-1}{1-0} = 2$ (b) $\frac{3^{a+1}-3^a}{a+1 - a} = \frac{3^a(3-1)}{1} = 2\cdot 3^a$

(c) $2 \cdot 3^a = 9\cdot 2$, $3^a = 9$, $a = 2$

8. (a) $k \approx 0.7$ (b) $k\cdot 2^1 \approx 1.4$ (c) $\frac{2-1}{1-0} = 1$

9. $f' : x \longrightarrow 3k(2)^x$. (a) $f'(0) = 3k \approx 2.1$, (b) $f'(4) = 48k$

≈ 33.6

10. (a) $e^{.32} \approx (1.350)(1.020) \approx 1.377$

(b) $f' : x \longrightarrow e$, $f'(0.3) = e^{0.3} \approx 1.350$

(c) $\frac{1.377 - 1.350}{0.32 - 0.30} = \frac{0.027}{0.02} = 1.35$

11. (a) $1 - (\frac{1}{2})^2 = \frac{3}{4}$

(b) $1 - \frac{15}{16} = 2^{-3/T}$; $2^{-4} = 2^{-3/T}$; $-4 = -\frac{3}{T}$; $T = \frac{3}{4}$ days.

12. $326 = 200\,(1 + \frac{7}{1200})^{84}$ $\qquad \frac{326}{200} = (1 + \frac{7}{1200})^{84}$

$x = 200\,(1 + \frac{7}{1200})^{168}$ $\qquad \frac{x}{200} = (1 + \frac{7}{1200})^{168}$

$\frac{x}{200} = (\frac{326}{200})^2$ $\qquad x = 200\,(1.63)^2 = 531$ Ans. $532

13. $f^{-1} : x \to \frac{x}{2}$ $\qquad g^{-1} : x \to \frac{x+1}{3}$ $\qquad (f \circ g) : x \to 6x-2$

(a) $(f \circ g)^{-1} : x \longrightarrow \frac{x+2}{6}$

(b) $f^{-1} \circ g^{-1} : x \to \frac{x+1}{6}$

(c) $g^{-1} \circ f^{-1} : x \to \frac{x+2}{6}$

14. f^{-1}. If $y = 1 - \sqrt{1-x^2}$, then $x^2 = y^2 - 2y$, and since the graph indicates $x \geq o$, $x = \sqrt{2y-y^2}$

Therefore f^{-1} is $f^{-1} : x \to \sqrt{2x-x^2}$, $o \leq x \leq 1$.

15. (a) $2^4 = 16$

(b) $\log_a x + \log_a y = \log_a xy = z$, so $a^z = xy$.

16. (a) $\log_4 \sqrt{2}^{\sqrt{2}} = \sqrt{2}\,\log_4 \sqrt{2} = \frac{1}{2}\sqrt{2}\log_4 2 = \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}$

$= \frac{1}{4}\sqrt{2}$

(b) $\log_{2x} \sqrt{2} = \frac{1}{3}$

$(2x)^{\frac{1}{3}} = \sqrt{2} = 2^{\frac{1}{2}}$

$2x = (2^{\frac{1}{2}})^3 = 2^{\frac{3}{2}}$

$x = 2^{\frac{1}{2}} = \sqrt{2}$

(c) $\log_x(3x^2) = 5$

$$x^5 = 3x^2$$

$$x^5 - 3x^2 = 0$$

$$x^2(x^3-3) = 0$$

The real roots are $x=0$ and $x=\sqrt[3]{3}$, and only $\sqrt[3]{3}$ fits in the original equation.

(a) $\ln a^2b = 2\ln a + \ln b = 11$

(b) $\ln(a/b^2) = \ln a - 2\ln b = -7$

(c) $(\ln a)(\ln b) = 3\cdot 5 = 15$

(d) $\ln a^{\ln b} = (\ln b)(\ln a) = 15$

(e) $\dfrac{\ln a}{\ln b} = \dfrac{3}{5}$

18. If $\log_4 x = y$, then $x = 4^y = 2^{2y}$, and $\log_2 x = 2y$.

Hence $\log_4 x = \frac{1}{2}\log_2 x$, and g is also $x \to \frac{1}{2}\log_2 x$.

(a) $(f\circ g)(3) = f(g(3)) = f(\frac{1}{2}\log_2 3) = 2^{\frac{1}{2}\log_2 3} = (2^{\log_2 3})^{\frac{1}{2}}$

$$= 3^{\frac{1}{2}} = \sqrt{3}\ .$$

(b) $(g\circ f)(3) = g(f(3)) = g(2^3) = \frac{1}{2}\log_2 2^3 = \frac{3}{2}$.

If $y = 2^x$, $x = \log_2 y$, and we have

$$f^{-1} : x \to \log_2(x)\ .$$

Similarly, if $y = \frac{1}{2}\log_2 x$, $x = 2^{2y}$, and we have

$$g^{-1} : x \to 2^{2x}$$

(c) $(f^{-1}\circ g^{-1})(3) = f^{-1}(2^6) = \log_2 2^6 = 6$

(d) $(g^{-1}\circ f^{-1})(3) = g^{-1}(\log_2 3) = 2^{2\log_2 3} = 3^2 = 9$

19. (a) $\log_6(x+4) + \log_6(x+3) = \log_6(x+4)(x+3) = 1$

$$(x+4)(x+3) = 6^1 = 6$$

$$x^2 + 7x + 12 = 6$$

$$x^2 + 7x + 6 = 0$$

$$(x+6)(x+1) = 0$$

$$x = -6\ ,\ -1$$

But neither of the original logs is defined if $x = -6$,

so we have only $x = -1$.

(b) $\log_{15} (2x+3)(x+1) = 1$

$$2x^2+5x+3 = 15^1 = 15$$

$$2x^2+5x-12 = 0$$

$$(2x-3)(x+4) = 0$$

$$x = \frac{3}{2}\ ;\ -4\ .$$

We reject the -4 , as above.

20. $2^{2x+2} + 2^{x+2} = 3$

Divide by $4 = 2^2$:

$$2^{2x} + 2^x = \frac{3}{4}$$

Set $y = 2^x$; then $y^2 = 2^{2x}$, and

$$y^2 + y = \frac{3}{4}$$

$$y^2 + y + \frac{1}{4} = 1$$

$$y + \frac{1}{2} = \pm 1$$

$$y = \frac{1}{2} \pm 1 = \frac{3}{2}\ ,\ -\frac{1}{2}$$

Since $y = 2^x$, we exclude $y = -\frac{1}{2}$.

The only root is then

$$x = \log_2 \frac{3}{2} = \log_2 3 - 1\ .$$

Chapter 5

CIRCULAR FUNCTIONS

Introduction

As mentioned in the introduction to this commentary, our treatment of circular functions assumes that the student has already completed a short unit of trigonometry. We assume that the student is familiar with the angle functions and their interrelationships; the simpler identities and equations; the trigonometry of triangles; the elementary facts about complex numbers; and the use of tables. We cover some of these topics again but so briefly that we cannot reasonably expect a student to master the material solely from this treatment.

The objective of this chapter is not to study the subject of trigonometry in all detail. Rather it is an attempt to give the student who has just studied polynomial, exponential and logarithmic functions some familiarity with a completely new kind of function, one which is periodic. In the process, some of the ideas involved in elementary trigonometry will become more meaningful through viewing them in a new light.

Section 5-1. Circular Motions and Periodicity

The emphasis throughout is on the periodic properties of the circular functions, i.e., the sine and cosine. In beginning the chapter you should emphasize that we shall talk here about functions which differ from those we have previously studied in that they have the property of periodicity.

One good way to visualize a periodic function is in terms of the machine developed in Section 1-13. If the function depicted by the machine is periodic, then when x, $x + a$, $x + 2a$, ..., $x + na$, are dropped into the hopper we obtain the same output $(f(x))$ in each case. In the next section we speak of laying rectangles containing one complete period of the function end to end and you may wish to use the idea here in order to illustrate further the meaning of periodicity.

The use of the u, v and x, y planes which we employ may be a source of difficulty at first. We wish to talk about the unit circle with which we define sin and cos, but later we shall need to display the graphs of $y = \sin x$ and $y = \cos x$ on an x, y plane. Since we are using x for arc length (to obtain the familiar $\sin x$ and $\cos x$) it would be easy to teach the student to visualize x as both the horizontal axis on the plane of the unit circle and at the same time as a length of circular arc. We feel that if care is exercised at the time the transition is made in Section 5-2, the use of u and v is more satisfactory than trying to get x to wear two hats in this section.

A more exact way of defining $\sin x$ and $\cos x$ is by a composition of two functions, one from the set of real numbers to the set of geometric points on the unit circle and the other from the set of points on the circle to the set of real numbers. Thus, if $x \in R$ and if p is a point on the unit circle, we have a function

$$f : x \longrightarrow p$$

and another function

$$g : p \longrightarrow \cos x$$

from which

$$f \circ g : x \longrightarrow \cos x$$

and similarly for the sine. We feel, however, that the way in which we have handled it in the text, while possibly less rigorous, is certainly easier to teach and is perfectly adequate for our purposes.

The fact that cosine and sine are real functions should be emphasized. You might point out to the student that nowhere in this section have we used an angle and although we have used the concept of arc length, sine and cosine are completely divorced from any geometric considerations. They are functions on the set of real numbers in the same sense as polynomials, say, or exponential functions. Too often when we speak of $\sin A$, the students feel that A must be an angle. Sometimes they think of A as being the degree measure or radian measure of an angle, but the idea that A need have no connection with an angle is usually very strange.

Exercises 5-1

The exercises lean on the notion of periodicity. The first 5 are not difficult. We have starred Exercises 6-9 since they require more insight than the others, but if Problem 7a is not assigned as homework, it should be covered in class, since this relationship is used in Section 5-4.

<u>Answers to Exercises 5-1</u>

1. The rotation of the earth about the sun every 365 days.

   The phases of the moon. Period is about $29\frac{1}{2}$ days.

   The swinging of the pendulum of a clock. For a grandfather's clock, the period is usually 2 seconds.

   The oscillation of a piston in a steam engine or internal combustion engine. Period depends upon speed of engine.

   The alternation of A.C. electric current. For 60 cycle current, the period is 1/60 second.

   Oscillation of vacuum tubes, vibrations of strings of musical instruments (sound waves in general), etc.

2. a) $P(-\frac{\pi}{2}) = P(\frac{3\pi}{2} - 2\pi) = P(\frac{3\pi}{2})$;

   b) $P(3\pi) = P(\pi + 2\pi) = P(\pi)$;

   c) $P(-\frac{3\pi}{2}) = P(\frac{\pi}{2} - 2\pi) = P(\frac{\pi}{2})$;

   d) $P(4076\pi) = P(0 + 2038 \times 2\pi) = P(0)$.

3. a) $(0, -1)$;

   b) $(-1, 0)$;

   c) $(0, 1)$;

   d) $(1, 0)$.

4. a) $x = \frac{3\pi}{2}, \frac{7\pi}{2}$

   b) $x = \pi, 3\pi$

   c) $x = 0, 2\pi$

   d) $x = \pi, 3\pi$

5. a) $x = \frac{\pi}{4}, \frac{5\pi}{4}$

   b) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

*6. a) $\sin 2x = \sin(2x + 2\pi)$ from periodicity of sin,

$= \sin 2(x + \pi)$ and the period is π.

b) $\sin \frac{1}{2}x = \sin\left(\frac{1}{2}x + 2\pi\right)$

$= \sin \frac{1}{2}(x + 4\pi)$ and the period is 4π.

c) $\cos 4x = \cos(4x + 2\pi)$

$= \cos 4(x + \frac{\pi}{2})$ and the period is $\frac{\pi}{2}$.

d) $\cos \frac{1}{2}x = \cos\left(\frac{1}{2}x + 2\pi\right)$

$= \cos \frac{1}{2}(x + 4\pi)$ and the period is 4π.

*7. a) $f(x) = f(x + a)$, $g(x) = g(x + a)$. Given.

$f(x) + g(x) = f(x + a) + g(x + a)$. Addition Axiom.

$(f + g)(x) = (f + g)(x + a)$. By definition.

$\therefore f + g$ is periodic with period a. By definition.

b) $f(x) \cdot g(x) = f(x + a) \cdot g(x + a)$ Multiplication Axiom.

$(f \cdot g)(x) = (f \cdot g)(x + a)$ Definition.

$\therefore f \cdot g$ is periodic with period a. Definition

*8. $f(x) = f(x + a)$ Given.

$g(x) = g(x)$ if g is defined at x.

$g(f(x)) = g(f(x + a))$ Substitution.

$(g \circ f)(x) = (g \circ f)(x + a)$ By definition of $g \circ f$.

$\therefore g \circ f$ is periodic with period a.

*9. By definition. $\cos : x \longrightarrow u = \cos x$

$\sin : x \longrightarrow v = \sin x$, where (u, v) is any point on the unit circle and x is arc length. If $x + a < x + 2\pi$, then

$(\cos(x + a), \sin(x + a)) \neq (\cos x, \sin x) = (u, v)$, since the coordinates (u, v) are unique.

5-2. Graphs of Sine and Cosine

The rectangle device used here can be a very useful one in teaching the student to graph periodic functions. By establishing the period and amplitude visually it directs his attention to a specific area of the plane with respect to both the domain and range of the function.

We use the geometric argument to obtain specific values of the functions, because it is the simplest and most familiar tool available to the student. We hope that you will emphasize the symmetric nature of the unit circle and that the student will be encouraged to use symmetry considerations whenever possible.

Exercises 5-2

The exercises develop some simple symmetric properties of $\sin x$ and $\cos x$, and lead the student into understanding the effect of the constants in $y = a \sin(bx + c)$.

Answers to Exercises 5-2

1. a) $f(3\pi) = f(\pi) = -1;$

   b) $f(\frac{7\pi}{3}) = f(\frac{\pi}{3}) = \frac{1}{2};$

   c) $f(\frac{9\pi}{2}) = f(\frac{\pi}{2}) = 0;$

   d) $f(\frac{25\pi}{6}) = f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2};$

   e) $f(-7\pi) = f(\pi) = -1;$

   f) $f(-\frac{10\pi}{3}) = f(\frac{2\pi}{3}) = -\frac{1}{2}.$

2. a) $f(\pi) = 0$;

b) $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$;

c) $f(\frac{\pi}{2}) = 1$;

d) $f(\frac{\pi}{6}) = \frac{1}{2}$;

e) $f(\pi) = 0$;

f) $f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$.

3. a) $x = \frac{\pi}{4} + 2n\pi, \ \frac{5\pi}{4} + 2n\pi$;

b) $x = \frac{3\pi}{4} + 2n\pi, \ \frac{7\pi}{4} + 2n\pi$;

c) $x = 0 + 2n\pi, \ \pi + 2n\pi \quad x = 2n\pi, \ (2n + 1)\pi$

d) For all values of x.

4. See Graphs.

5. See Graphs.

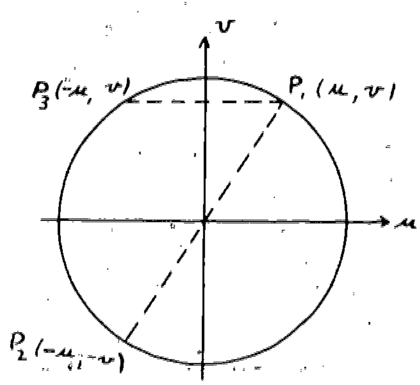
6. See Graphs.

7. a) The values of the ordinate are multiplied by k.

b) The period of the graph is $\frac{2\pi}{k}$.

c) The graph is shifted to the left by the amount $x = k$.

8. $\cos(x - \frac{\pi}{2}) = \sin x$.

9. 

a) P_1 and P_2 are symmetric with respect to the origin.

$P_1 = P(x) = (u, v)$,

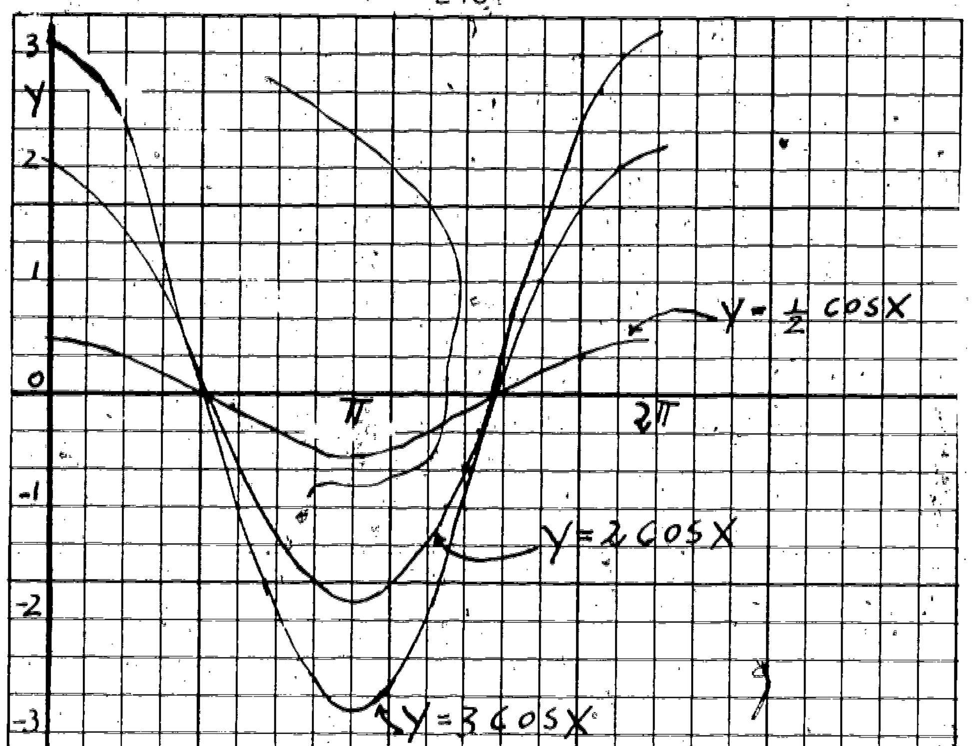
$P_2 = P(x - \pi) = P(x + \pi)$

$= (-u, -v)$.

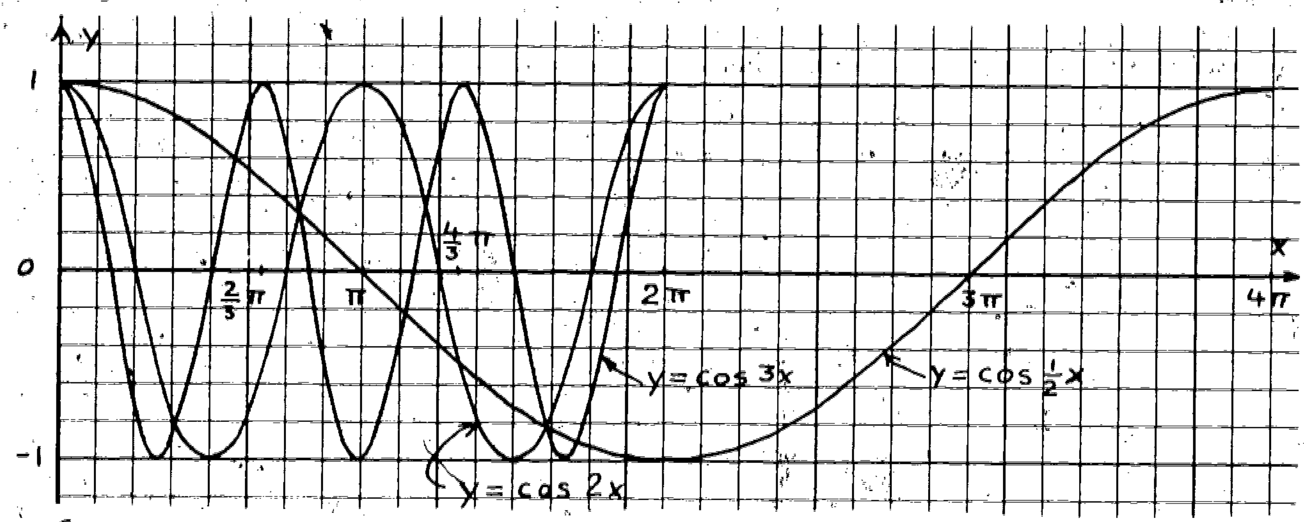
Hence, $\cos x = -\cos(x - \pi) = -\cos(x + \pi)$, and

$\sin x = -\sin(x - \pi) = -\sin(x + \pi)$.

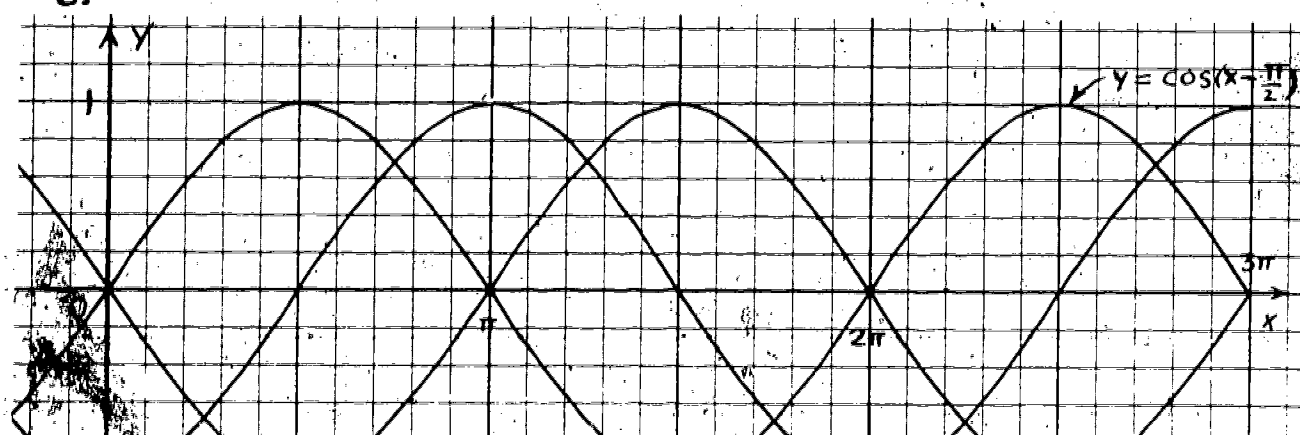
4.


y
3
2
1
0
-1
-2
-3
π
2π
y = ½ cos x
y = 2 cos x
y = 3 cos x

5.


y
1
0
-1
x
2/3 π
π
4/3 π
2π
3π
4π
y = cos 3x
y = cos ½ x
y = cos 2x

6.


y
1
-1
x
π
2π
3π
y = cos(x − π/2)
y = cos(x + π)
y = cos(x + π/2)

9\. b) P_1 and P_3 are symmetric with respect to the v-axis.

$P_1 = P(x) = (u, v)$,

$P_2 = P(-x - \pi) = P(-x + \pi) = (-u, v)$.

Hence, $\cos x = -\cos(-x - \pi) = -\cos(-x + \pi)$,

and $\sin x = \sin(-x - \pi) = \sin(-x + \pi)$.

5-3. Angle and Angle Measure

This is probably review material for most students at this level. Formulas (1) and (2) are the standard radian-degree relationships and the exercises are routine drill in going from one to the other. The paragraph devoted to the relationship between the angle and the area in its included sector will be used later to evaluate the limit of $\frac{\sin x}{x}$ as x approaches 0.

Answers to Exercises 5-3

1\. a) 120° d) 210° g) 480°

b) 30° e) 360° h) 648°

c) -120° f) 150° i) 585°

2\. a) $\frac{3\pi}{2}$ d) $\frac{8\pi}{3}$ g) $\frac{9\pi}{2}$

b) $-\frac{\pi}{6}$ e) $\frac{13\pi}{12}$ f) $\frac{19\pi}{18}$

c) $\frac{3\pi}{4}$ f) $-\frac{7\pi}{12}$ i) $\frac{\pi}{10}$

3\. $\alpha = \frac{2A}{r^2} = \frac{2 \cdot 9\pi}{9/4} = 8\pi$

4\. $A = \frac{\alpha r^2}{2} = \frac{(3/2)\pi \cdot 4}{2} = 3\pi$ square units.

5. a) Since $90^\circ = 100$ "units", $1^\circ = \frac{10}{9}$ "units."

   b) Since $\frac{\pi}{2} = 100$ "units", 1 radian $= \frac{200}{\pi}$ "units."

   c) $\alpha = \frac{s}{r} = \frac{2r}{r} = 2$ radians; hence $\alpha = \frac{400}{\pi}$ "units."

5-4. Uniform Circular Motion

This unit should be taught with care, since the material included will be used in Section 5-8. In dealing with sin and cos as time functions we use ωt where ω is the angular velocity, because this is the form in which it appears in most scientific applications. Since up to this point we have dealt with functions connected with an arc length x, you should spend a little time familiarizing the student with ωt.

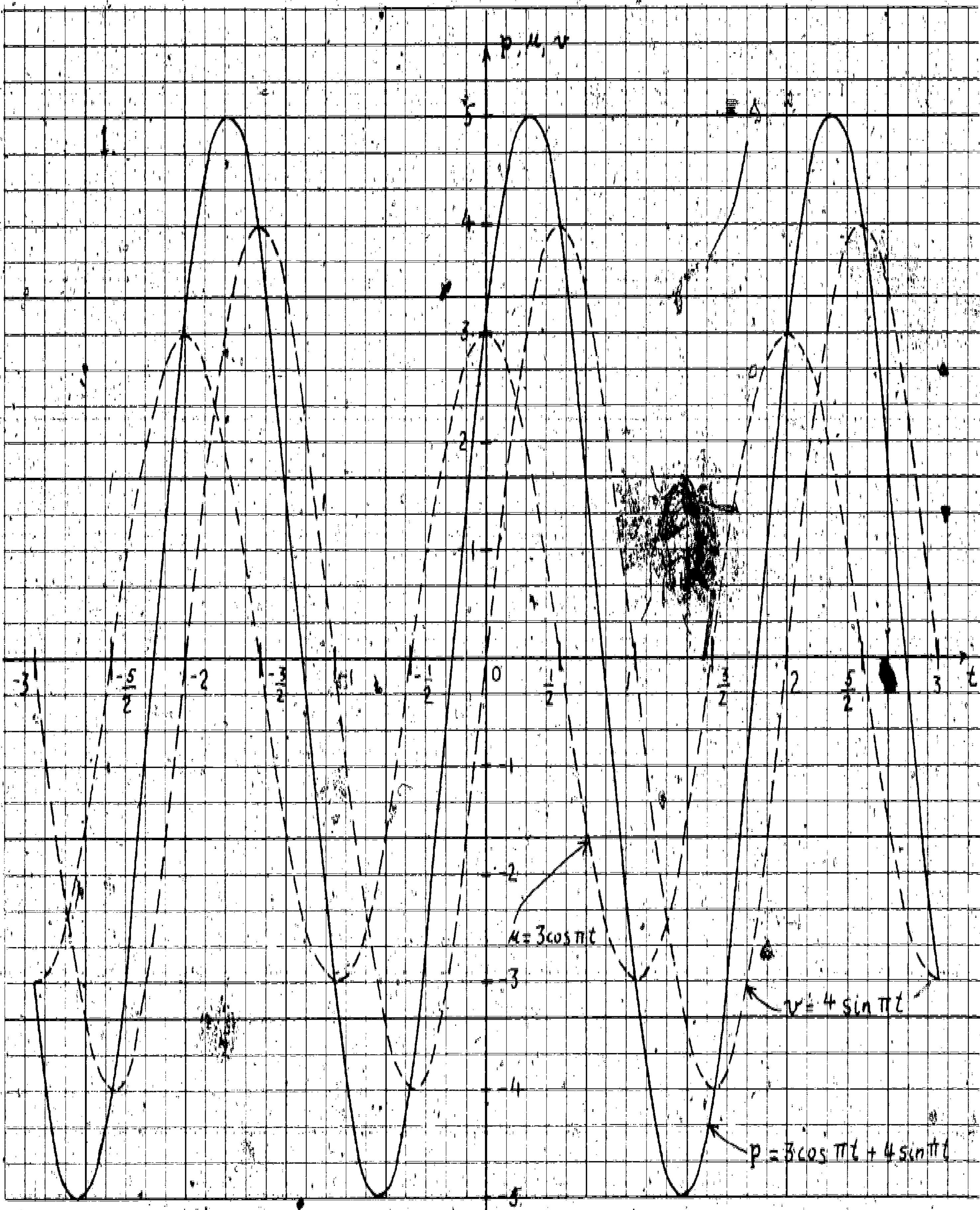
The device used in the text to visualize the behavior of a wave is only one of several which you may wish to try. Most currently available trigonometry texts have some such approach to the problem, and you should supplement the textual explanation with any other means you feel appropriate.

We chose the acoustical example to build upon since the addition of pressures is intuitively simple.

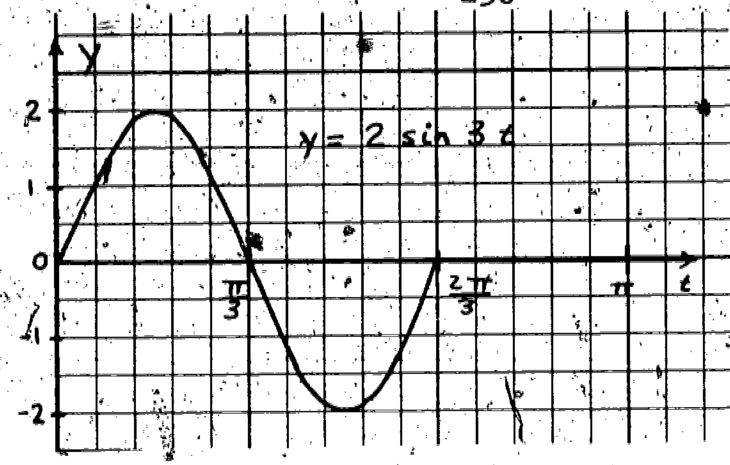
Answers to Exercises 5-4

1. See Graph.

   The graph of $p = 3 \cos \pi t + 4 \sin \pi t$ is periodic, with period 2, since corresponding points on the graph are 2, 4, 6, ... , 2n units apart when measured along the t-axis. (Periods of $3 \cos \pi t$ and $4 \sin \pi t$ same as period of p.)


p, u, v
t
$u = 3\cos \pi t$
$v = 4\sin \pi t$
$p = 3\cos \pi t + 4\sin \pi t$

2. a)

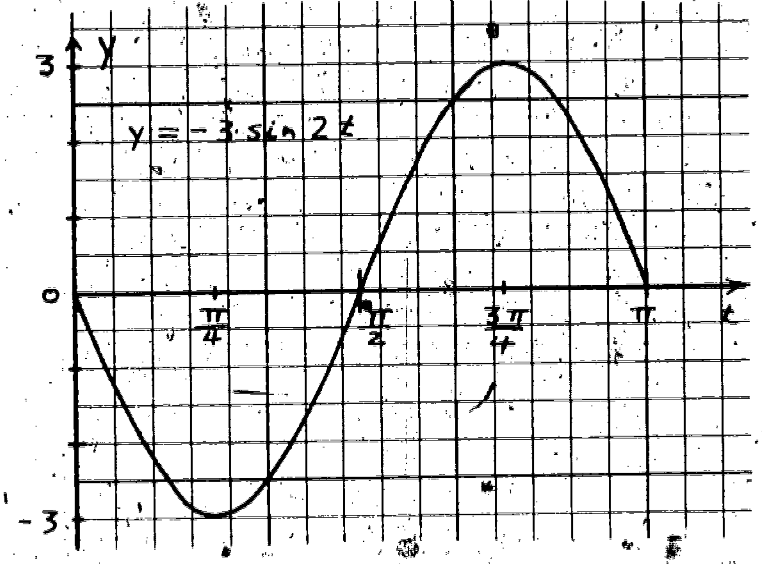


The period is $\frac{2\pi}{3}$.

The range is

$2 \leq y \leq 2$.

b)

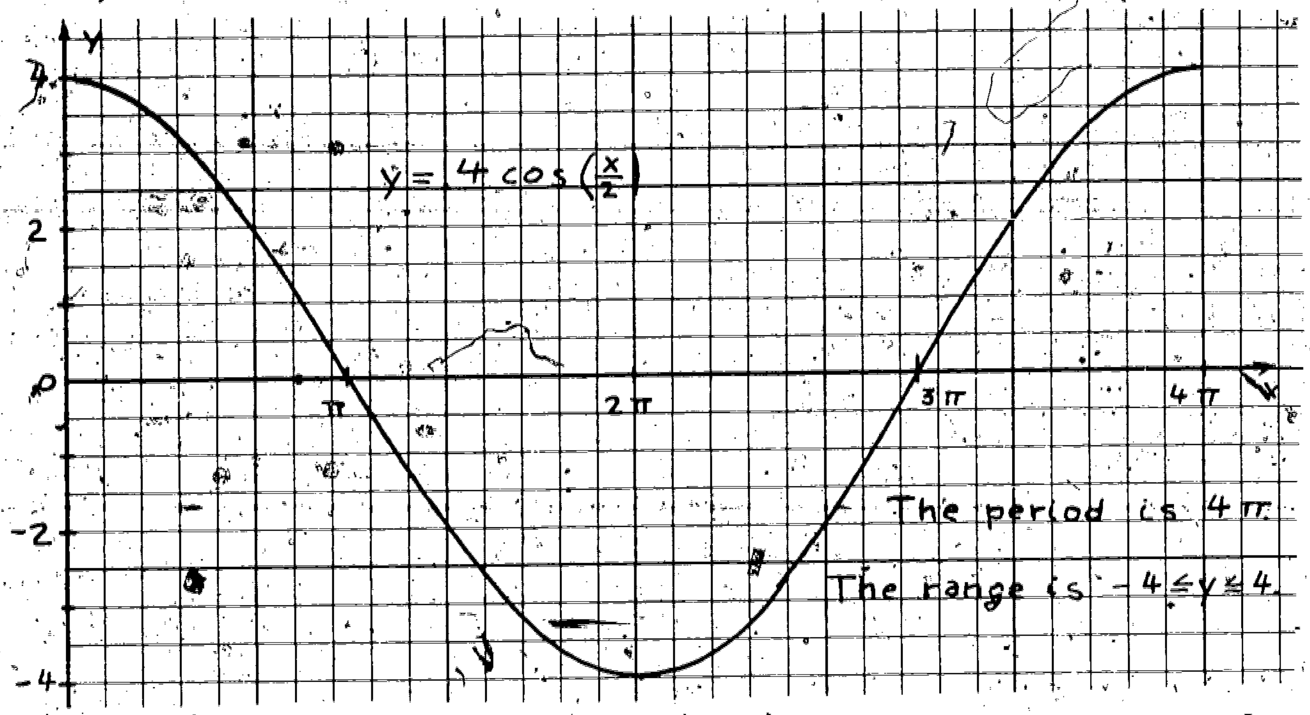


The period is π.

The range is

$-3 \leq y \leq 3$.

c)

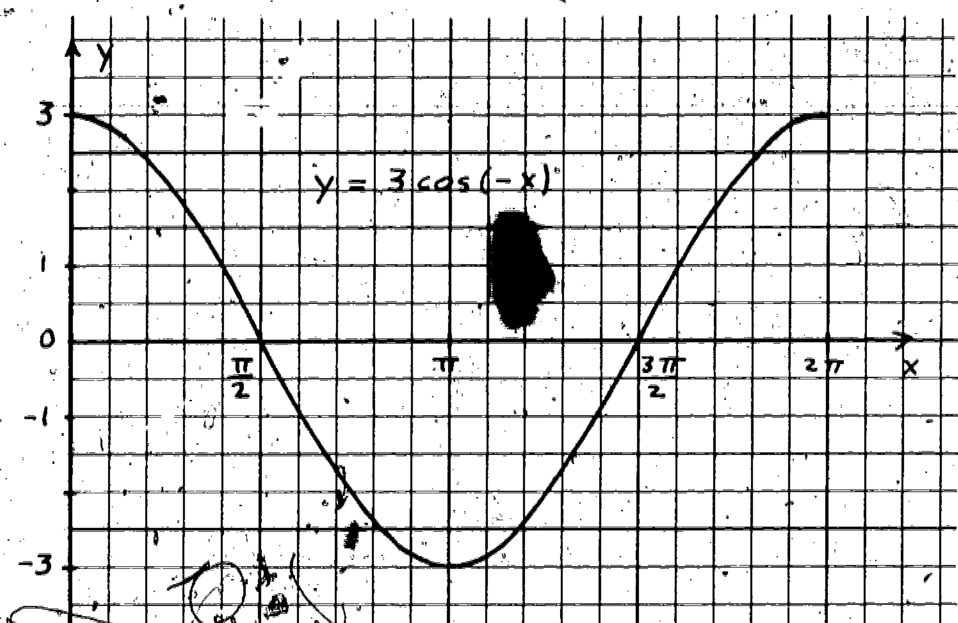


The period is 4π.

The range is $-4 \leq y \leq 4$.

2.

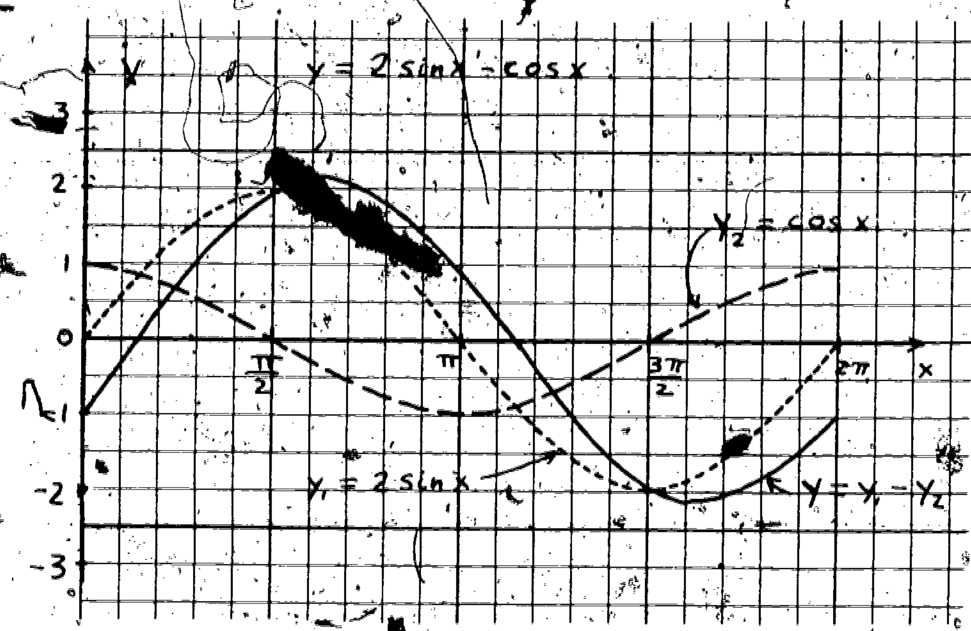
d)



The period is 2π.

The range is $-3 \le y \le 3$.

e)



The period is 2π.

The range is approximately $-2.2 \le y \le 2.2$

2. See Graphs.

5-5. Vectors and Rotations

We chose the vector approach to the addition formulas for two reasons. First, it should be a different means of deriving these relationships than that which the student has previously encountered. Second, it is an extremely simple and efficient means of obtaining these relationships. We do not intend this to be a thorough treatment of vectors, and we confine our attention to vectors of length ≤ 1, since this is sufficient for our purpose.

We anticipate that the use of the operator notation $R_x(R_{x'}(U))$ etc., will have to be explained very carefully since it is something which most of the students will never have encountered before. You should do a lot of blackboard work here, giving a variety of simple manipulative illustrations. Rotate vectors in both directions; illustrate rotations followed by rotations; show the rotations of the components of the vector as the vector rotates; in general, make sure that the ideas involved and the symbolism expressing the ideas are clear.

Exercises 5-5

You may wish to devise additional drill exercises in the use of rotation. Exercises 1, 2, 3 and 4 are cases in point and such problems are easy to make up.

Answers to Exercises 5-5

1. $T = (\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V,$

   $x = \frac{\pi}{4}$

2. a) $T = (-\frac{1}{2})U + (\frac{\sqrt{3}}{2})V,$

   $x = \frac{2\pi}{3}$.

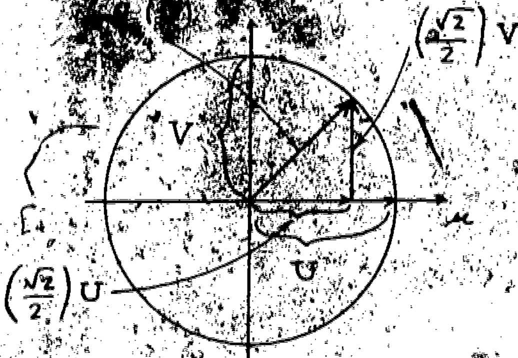
   b) $T = (-\frac{\sqrt{3}}{2})U + (-\frac{1}{2})V,$

   $x = \frac{7\pi}{6}$.

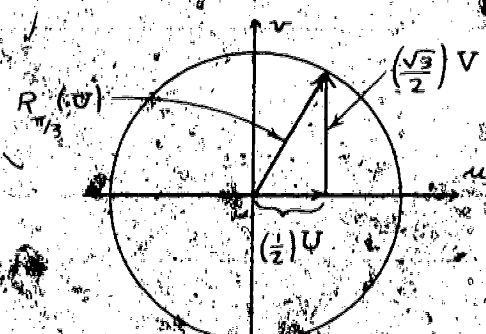
3. $R_{3\pi/2}(U) = (0)U + (-1)V = -V,$

   $R_{2\pi}(U) = U$

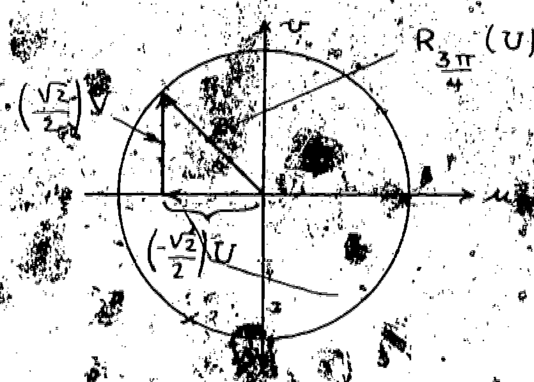
4. a) $R_{\pi/4}(U) = (\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V.$



   b) $R_{\pi/3}(U) = (\frac{1}{2})U + (\frac{\sqrt{3}}{2})V.$



5. $R_{3\pi/4}(U) = (-\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V.$



6. $R_{3\pi/4}(U) = R_{\pi/4+\pi/2}(U)$

$= R_{\pi/4}(R_{\pi/2}(U))$ (by Equation 7)

$= R_{\pi/4}(V)$.

7. Left member: $R_{x'}(R_x(U)) = R_{x'+x}(U)$;

Right member: $R_x(R_{x'}(U)) = R_{x+x'}(U)$.

Since x and x' are real numbers, $x' + x = x + x'$, and it follows that the left member equals the right member.

Geometrically, the result means that a rotation through arc x followed by a rotation through arc x' is equivalent to a rotation through arc x' followed by a rotation through arc x.

7. Alternate solution.

$R_{x'}(R_x(U)) = R_{x'+x}(U) = R_{x+x'}(U) = R_x(R_{x'}(U))$.

8. Since $V = R_{\pi/2}(U)$,

$R_x(V) = R_x(R_{\pi/2}(U))$

$= R_{\pi/2}(R_x(U))$, by the result of Exercise 7.

9. From Exercise 8, $R_x(V) = R_{\pi/2}(R_x(U))$

$$
\begin{aligned}
R_x(V) &= R_{\pi/2}(R_x(U)) \\
&= R_{\pi/2}(T_1 + T_2) \\
&= R_{\pi/2}(T_1) + R_{\pi/2}(T_2) \\
&= R_{\pi/2}(uU) + R_{\pi/2}(vV) \\
&= uR_{\pi/2}(U) + vR_{\pi/2}(V) \\
&= uV + (-v)U \\
&= uV - vU
\end{aligned}
$$

5-6 Addition Formulas for Sine and Cosine

The derivation of $\cos(x + x')$ and $\sin(x + x')$ is usually accomplished either by geometric considerations in the first quadrant (which then involve a great deal of work to generalize), or by use of the distance formula. As remarked before, we feel the vector approach to be new and instructive and, in essence, simpler than either of the aforementioned. We include the page on the relation to complex numbers to show still another means of deriving them.

Exercises 5-6

The exercises are, in general, identities; applications of the sum and difference formulas. You may wish to illustrate a few samples on the blackboard before asking the students to work the exercises. Exercises 4, 5 and 6 are important since the tangent function appears here for the first time and some of its properties are investigated. You should be sure to cover these exercises at some point in the work.

Answers to Exercises 5-6

1\. a) $\cos(\frac{\pi}{2} - x) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 + \sin x$

$= \sin x$

b) $\sin(\frac{\pi}{2} - x) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$= \cos x - 0$

$= \cos x$

c) $\cos(x + \frac{\pi}{2}) = \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2}$

$= 0 - \sin x$

$= -\sin x$

d) $\sin(x + \frac{\pi}{2}) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2}$

$= 0 + \cos x$

$= \cos x$

e) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$= (-1)\cos x + 0$

$= -\cos x$

f) $\sin(\pi - x) = \sin \pi \cos x - \cos \pi \sin x$

$= 0 - (-1) \sin x$

$= \sin x$

g) $\cos\left(\frac{3\pi}{2} + x\right) = \cos \frac{3\pi}{2} \cos x - \sin \frac{3\pi}{2} \sin x$

$= 0 - (-1) \sin x$

$= \sin x$

h) $\sin\left(\frac{3\pi}{2} + x\right) = \sin \frac{3\pi}{2} \cos x + \cos \frac{3\pi}{2} \sin x$

$= (-1) \cos x + 0$

$= -\cos x$

i) $\sin\left(\frac{\pi}{4} + x\right) = \sin \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \sin x$

$= \left(\frac{\sqrt{2}}{2}\right)(\cos x + \sin x);$

$\cos\left(\frac{\pi}{4} - x\right) = \cos \frac{\pi}{4} \cos x + \sin \frac{\pi}{4} \sin x$

$= \left(\frac{\sqrt{2}}{2}\right)(\cos x + \sin x).$

Hence, $\sin\left(\frac{\pi}{4} + x\right) = \cos\left(\frac{\pi}{4} - x\right).$

2. $\sin(x - x') = \sin[x + (-x')]$

$= \sin x \cos(-x') + \cos x \sin(-x')$

$= \sin x \cos x' - \cos x \sin x'$

*3. Formula (9): $\cos(x - x') = \cos x \cos x' + \sin x \sin x'$

To derive (6): $\cos(x + x') = \cos[x - (-x')]$

$= \cos x \cos(-x') + \sin x \sin(-x')$

$= \cos x \cos x' - \sin x \sin x'$

To derive (7): $\sin(x + x') = \cos\left[\frac{\pi}{2} - (x + x')\right]$ from Exercise 1(a).

$= \cos\left[\left(\frac{\pi}{2} - x\right) - x'\right]$

$= \cos\left(\frac{\pi}{2} - x\right) \cos x'$

$+ \sin\left(\frac{\pi}{2} - x\right) \sin x'$

To simplify $\cos\left(\frac{\pi}{2} - x\right)$ and $\sin\left(\frac{\pi}{2} - x\right)$, use Exercise 1a.

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right]$$
$$= \cos x.$$

Hence, $\cos(\frac{\pi}{2} - x) \cos x' + \sin(\frac{\pi}{2} - x) \sin x'$

becomes $\sin x \cos x' + \cos x \sin x'$.

Therefore, $\sin(x + x') = \sin x \cos x' + \cos x \sin x'$.

To derive (10), use (7) just obtained:

$$\begin{aligned}\sin(x - x') &= \sin[x + (-x')] \\ &= \sin x \cos(-x') + \cos x \sin(-x') \\ &= \sin x \cos x' - \cos x \sin x'.\end{aligned}$$

4. $\tan : x \to \dfrac{\sin x}{\cos x} \quad (x \neq \pm \frac{\pi}{2} \pm 2n\pi)$

To prove that tan is periodic with period π, we must prove that $\tan(x + \pi) = \tan x$.

From the definition, $\tan(x + \pi) = \dfrac{\sin(x + \pi)}{\cos(x + \pi)}$

$= \dfrac{-\sin x}{-\cos x}$ (from Exercise 5-2, 9a)

$= \tan x$

Now $\tan(\pm \frac{\pi}{2} \pm 2n\pi) = \dfrac{\sin(\pm \frac{\pi}{2} \pm 2n\pi)}{\cos(\pm \frac{\pi}{2} \pm 2n\pi)}$.

But the denominator of this fraction in zero and therefore the values of $\tan(\pm \frac{\pi}{2} \pm 2n\pi)$ are undefined.

5. $\tan(x \pm x') = \dfrac{\sin(x \pm x')}{\cos(x \pm x')}$

$= \dfrac{\sin x \cos x' \pm \cos x \sin x'}{\cos x \cos x' \mp \sin x \sin x'}$

Dividing numerator and denominator by $\cos x \cos x'$,

$$= \frac{\frac{\sin x \cos x'}{\cos x \cos x'} \pm \frac{\cos x \sin x'}{\cos x \cos x'}}{\frac{\cos x \cos x'}{\cos x \cos x'} \mp \frac{\sin x \sin x'}{\cos x \cos x'}}$$

$$= \frac{\frac{\sin x}{\cos x} \pm \frac{\sin x'}{\cos x'}}{1 \mp \frac{\sin x}{\cos x} \cdot \frac{\sin x'}{\cos x'}}$$

$$= \frac{\tan x \pm \tan x'}{1 \mp \tan x \tan x'}$$

6. $\tan (\pi - x) = \frac{\tan \pi - \tan x}{1 + \tan \pi \tan x}$

$$= \frac{0 - \tan x}{1 + 0}$$

$$= - \tan x$$

$\tan (\pi + x) = \frac{\tan \pi + \tan x}{1 - \tan \pi \tan x}$

$$= \frac{0 + \tan x}{1 - 0} = \tan x$$

$\tan(- x) = \frac{\sin (-x)}{\cos (-x)} = \frac{- \sin x}{\cos x} = - \tan x$

7. $\sin 2x = \sin (x + x) = \sin x \cos x + \cos x \sin x$

$$= 2 \sin x \cos x$$

$\cos 2x = \cos (x + x) = \cos x \cos x - \sin x \sin x$

$$= \cos^2 x - \sin^2 x$$

$\tan 2x = \tan (x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$

$$= \frac{2 \tan x}{1 - \tan^2 x}$$

8. $\sin 3x = \sin(2x + x)$

$$= \sin 2x \cos x + \cos 2x \sin x$$

$$= 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x$$

$$= 3 \sin x \cos^2 x - \sin^3 x$$

9. $\cos 2x = 1 - 2 \sin^2 x$

Let $x = \frac{y}{2}$

$$\cos y = 1 - 2 \sin^2 \frac{y}{2}$$

$$\sin^2 \frac{y}{2} = \frac{1 - \cos y}{2}$$

$$\sin \frac{y}{2} = \pm\sqrt{\frac{1 - \cos y}{2}}$$

10. $\cos 2x = 2 \cos^2 x - 1$

Let $x = \frac{y}{2}$

$$\cos y = 2 \cos^2 \frac{y}{2} - 1$$

$$\cos^2 \frac{y}{2} = \frac{1 + \cos y}{2}$$

$$\cos \frac{y}{2} = \pm\sqrt{\frac{1 + \cos y}{2}}$$

11. $\tan \frac{y}{2} = \dfrac{\sin \frac{y}{2}}{\cos \frac{y}{2}}$

$$= \pm\sqrt{\frac{1 - \cos y}{1 + \cos y}}$$

Multiply the right member by $\sqrt{\frac{1 - \cos y}{1 - \cos y}}$;

$$\tan \frac{y}{2} = \pm\sqrt{\frac{(1 - \cos y)^2}{1 - \cos^2 y}}$$

$$= \pm\sqrt{\frac{(1 - \cos y)^2}{\sin^2 y}}$$

$$= \frac{1 - \cos y}{\sin y}$$

Note: The result given is correct since $\tan \frac{4}{2}$ and $\frac{1 - \cos y}{\sin y}$ agree in sign in all four quadrants.

11. (cont'd) Multiply the right member by $\sqrt{\frac{1 + \cos y}{1 + \cos y}}$;

$$\tan \frac{y}{2} = \pm \sqrt{\frac{1 - \cos^2 y}{(1 + \cos y)^2}}$$

$$= \pm \sqrt{\frac{\sin^2 y}{(1 + \cos y)^2}}$$

$$= \frac{\sin y}{1 + \cos y}$$ (See previous note.)

12. $R_x(U) = [\cos x + i \sin x]\ 1$

$$R_{\pi/2}(U) = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$= 0 + i$$

$$= i = V$$

$$R_{\pi}(U) = \cos \pi + i \sin \pi$$

$$= -1 + 0$$

$$= -1 = -U$$

13. $R_{\pi/2}(R_x(U)) = R_{\pi/2}(\cos x\, U + \sin x\, V)$

Rotation through $\pi/2$ is equivalent to multiplication by i.

So, $i(\cos x + i \sin x) = i(\cos x\, U + \sin x\, V)$

$$-\sin x + i \cos x = i[\cos x(1) + \sin x(i)]$$

$$-\sin x + i \cos x = -\sin x + i \cos x$$

$$R_x(V) = (\cos x + i \sin x)\ V$$

$$= (\cos x + i \sin x)\ i$$

$$= -\sin x + i \cos x$$

$$= -\sin x\, U + \cos x\, V.$$

5-7 Construction and Use of Tables of Circular Functions

Since this material is largely in the nature of a review, you will probably not wish to spend much time on it. The table of decimal fractions of $\pi/2$ will be new to the student, but we use it as we do any other table and it should cause no difficulty.

Answers to Exercises 5-7a.

1. Table I is not folded because the values of x are given in such a way that they are not symmetrical about $x = \frac{\pi}{4} \approx 0.785$. For example,

   $$\cos 0.60 = \sin(\tfrac{\pi}{2} - 0.60).$$ Since we are using radian measure, $\frac{\pi}{2}$ is irrational, and hence we would have to use an irrational interval (as is done in Table II) to get a symmetric table.

   $$\cos 0.60 \approx \sin(1.57 - 0.60)$$
   $$\approx \sin 0.97.$$

   From the table, $\cos 0.60 = 0.8253$ and $\sin 0.97 = 0.8249$. To be able to fold the table, the values of $\cos 0.60$ and $\sin 0.97$ would have to be the same.

2. a) $\sin 0.73 \approx 0.6669$, $\cos 0.73 \approx 0.7452$

   b) $\sin(-5.17) = \sin(-5.17 + 2\pi)$
   $$\approx \sin 1.11$$
   $$\approx 0.8957$$

   $\cos(-5.17) \approx \cos 1.11 \approx 0.4447$

   c) $\sin 1.55 \approx 0.9998$, $\cos 1.55 \approx 0.0208$

2. d) $\sin 6.97 = \sin(6.97 - 2\pi)$

$\approx \sin 0.69 \approx 0.6365$

~~$\cos 6.97 \approx \cos 0.69 \approx 0.7712$~~

3. a) $\sin x \approx 0.1099$, $x \approx 0.11$

b) $\cos x \approx 0.9131$, $x \approx 0.42$

c) $\sin x \approx 0.6495$, $x \approx 0.71$

d) $\cos x \approx 0.5403$, $x \approx 1.00$

Note: Hereafter we use = for ≈.

4. a) $\sin 0.31(\frac{\pi}{2}) = 0.468$, $\cos 0.31(\frac{\pi}{2}) = 0.884$

b) $\sin 0.79(\frac{\pi}{2}) = 0.946$, $\cos 0.79(\frac{\pi}{2}) = 0.324$

c) $\sin 0.62(\frac{\pi}{2}) = 0.827$, $\cos 0.62(\frac{\pi}{2}) = 0.562$

d) $\sin 0.71(\frac{\pi}{2}) = 0.898$, $\cos 0.71(\frac{\pi}{2}) = 0.440$

5. a) $\sin \omega t = 0.827$, $t = 0.62$

b) $\cos \omega t = 0.905$, $t = 0.28$

c) $\sin \omega t = 0.475$, $t = 0.315$

d) $\cos \omega t = 0.795$, $t = 0.425$

6. a) $\sin 45^\circ = 0.7071$, $\cos 45^\circ = 0.7071$

b) $\sin 73^\circ = 0.9563$, $\cos 73^\circ = 0.2924$

c) $\sin 36.2^\circ = 0.5906$, $\cos 36.2^\circ = 0.8069$

d) $\sin 81.5^\circ = 0.9890$, $\cos 81.5^\circ = 0.1478$

7. a) $\sin x = 0.6293$, $x = 39^\circ$

b) $\cos x = 0.9914$, $x = 7.5^\circ$

c) $\sin x = 0.6214$, $x = 38.4^\circ$

d) $\cos x = 0.8949$, $x = 26.5^\circ$

Answers to Exercises 5-7b

1. $\sin 1.73 = \sin(\pi - 1.73) = \sin 1.41 = 0.9871$ (Table I)

2. $\cos 1.3\pi = -\cos(1.3\pi - \pi) = -\cos 0.3\pi = -\cos 0.60(\frac{\pi}{2})$
$= -0.588$ (Table II)

3. $\sin(-.37) = -\sin .37 = -0.3616$ (Table I)

4. $\sin(-.37\pi) = -\sin .74(\frac{\pi}{2}) = -0.918$ (Table II)

5. $\cos 2.8\pi = \cos(2.8\pi - 2\pi) = \cos 0.8\pi = -\cos(\pi - 0.8\pi)$
$= -\cos 0.2\pi = -\cos 0.4(\frac{\pi}{2}) = -0.809$ (Table II)

6. $\cos 1.8\pi = \cos(2\pi - 1.8\pi) = \cos 0.2\pi = 0.809$ (from Ex. 5)

7. $\cos 3.71 = -\cos(3.71 - \pi) = -\cos 0.57 = -0.8419$ (Table I)

8. $\sin 135^\circ = \sin(180^\circ - 135^\circ) = \sin 45^\circ = 0.7071$ (Table III)

9. $\cos(-135^\circ) = -\cos(180^\circ - 135^\circ) = -\cos 45^\circ = -0.7071$
(Table III)

10. $\sin 327^\circ = -\sin(360^\circ - 327^\circ) = -\sin 33^\circ = -0.5446$
(Table III)

11. $\cos(-327^\circ) = \cos(360^\circ - 327^\circ) = \cos 33^\circ = 0.8387$ (Table III)

12. $\cos 12.4\pi = \cos(12.4\pi - 12\pi) = \cos 0.4\pi = \cos 0.8(\frac{\pi}{2})$
$= 0.309$ (Table II)

13. $\sin 12.4 = -\sin(4\pi - 12.4) = -\sin 0.16 = -0.1593$ (Table I)

*14. $\cos(\sin .3\pi) = \cos(\sin 0.6(\frac{\pi}{2})) = \cos 0.809$ (Table II)
$= 0.6902$ (Table I)

*15. $\sin(\sin .7) = \sin 0.6442 = 0.6006$ (Table I)

5-8. Pure Waves. Frequency, Amplitude and Phase

We chose cos as our standard wave because its first peak occurs at 0. Since we are using peaks to discuss phase, cos serves better than sin which peaks first at $\pi/2$. By using

$0 \leq \alpha < 2\pi$, we avoid all mention of a wave "leading". This is a departure from the convention that most sciences which have occasion to discuss lead and lag use both. You may wish to explore this idea by examining with the class the effect of using $-\pi \leq \alpha < \pi$, and show that $\alpha < 0$ represents a lead in the sense that $\alpha > 0$ represents a lag.

The problem of locating the zeros of

$$p = 3 \cos \pi t + 4 \sin \pi t$$

is mentioned in the text and dropped. The problem works out as follows:

Since $\quad p \approx 5 \cos(\pi t - 0.927)$,

$$p = 0 \text{ if } \pi t - 0.927 \approx \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$

Hence, if

$$t \approx \frac{0.927}{\pi} + \frac{1}{2} \text{ or } \frac{0.927}{\pi} + \frac{3}{2}.$$

Since $\frac{0.927}{\pi} \approx 0.29$, $\quad t \approx 0.79$ or 1.79.

Answers to Exercises 5-8

1. Equation (4) : $y = 3 \cos \pi t + 4 \sin \pi t$

$$\approx 5 \cos(\pi t - 0.927)$$

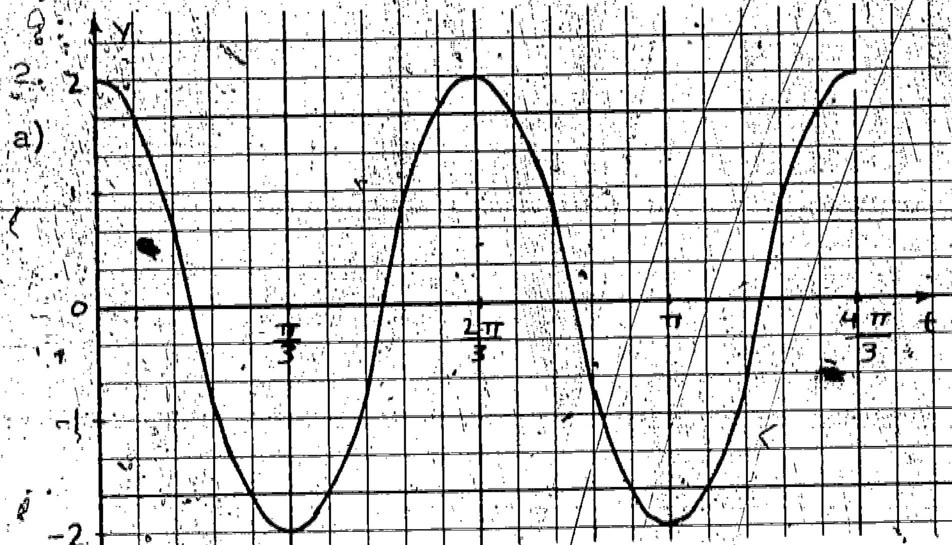
y will reach a minimum when $\pi t - 0.927 = \pi$

$$t = \frac{\pi + 0.927}{\pi} \approx 1.29$$

This agrees with the data shown in Figure 5-4e.

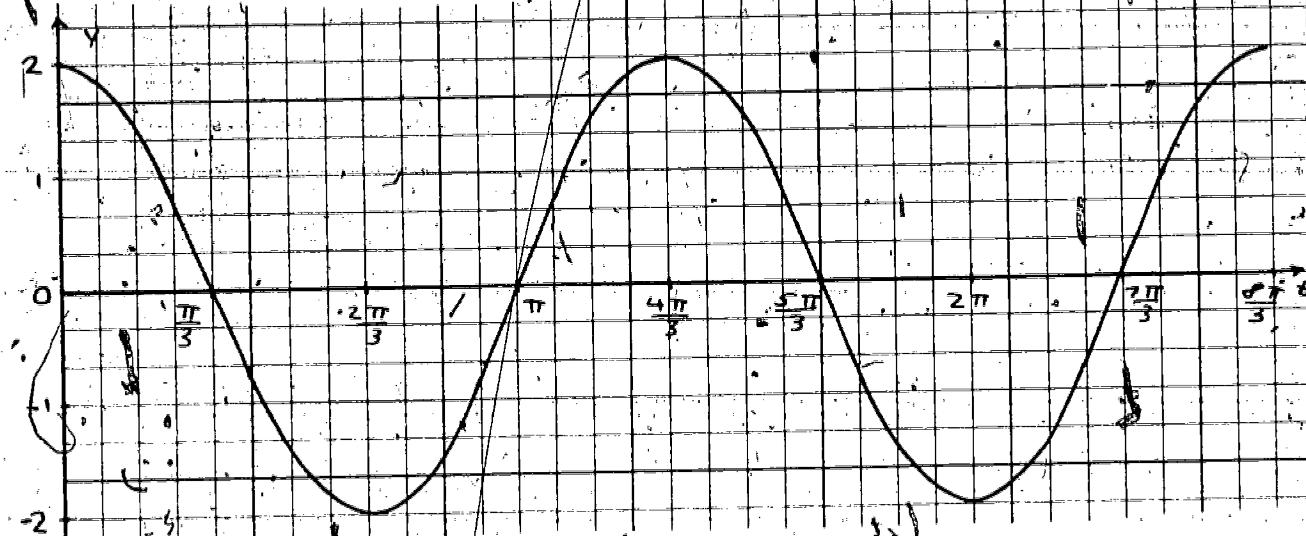
2. See graphs.

a)



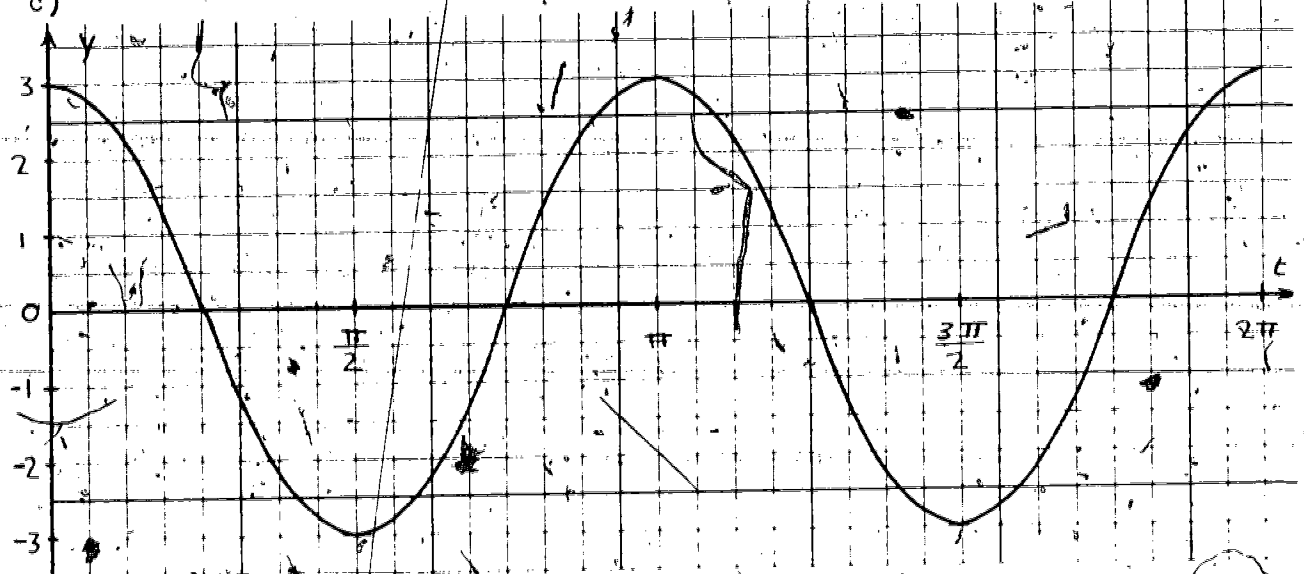
Amplitude = 2, Period = $\frac{2\pi}{3}$, Phase = 0.

b)



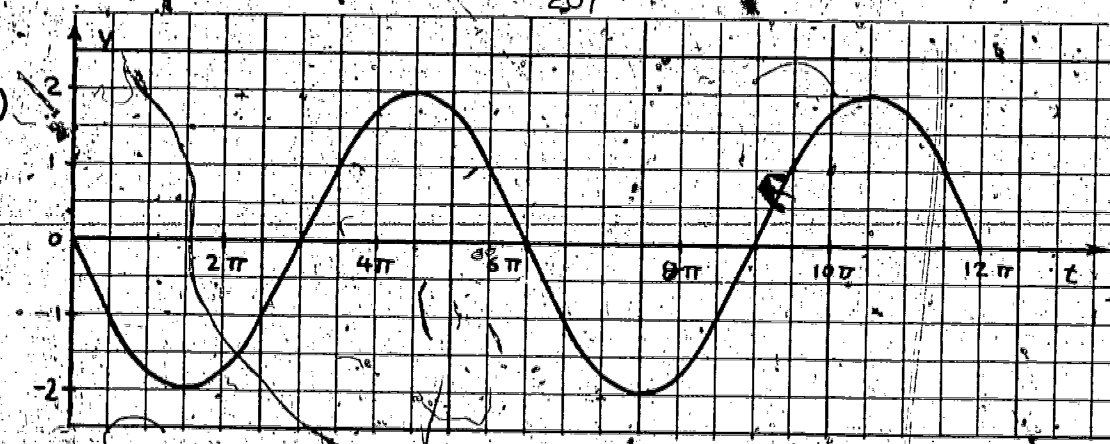
Amplitude = 2, Period = $4\pi/3$, Phase = 0.

c)



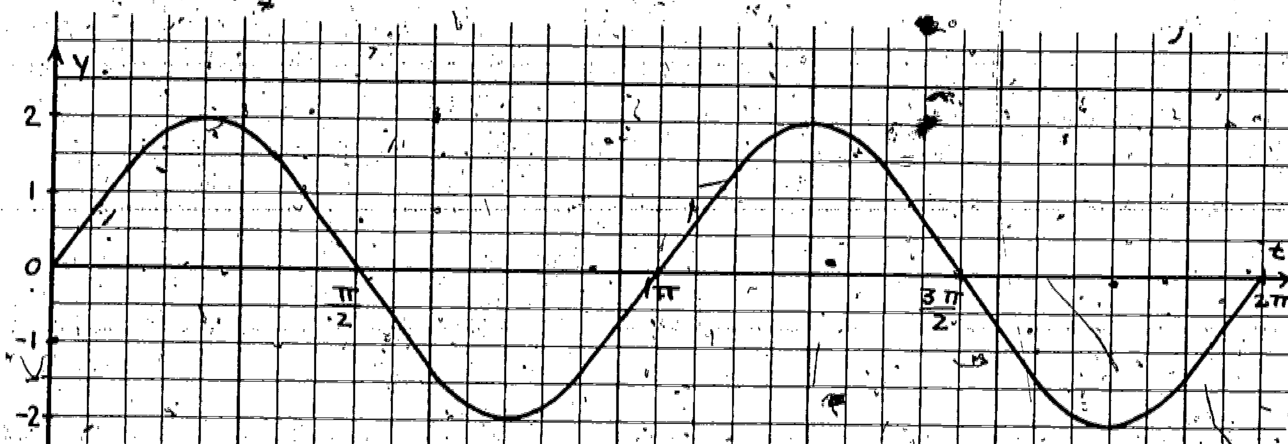
Amplitude = 3, Period = π, Phase = 0.

d)



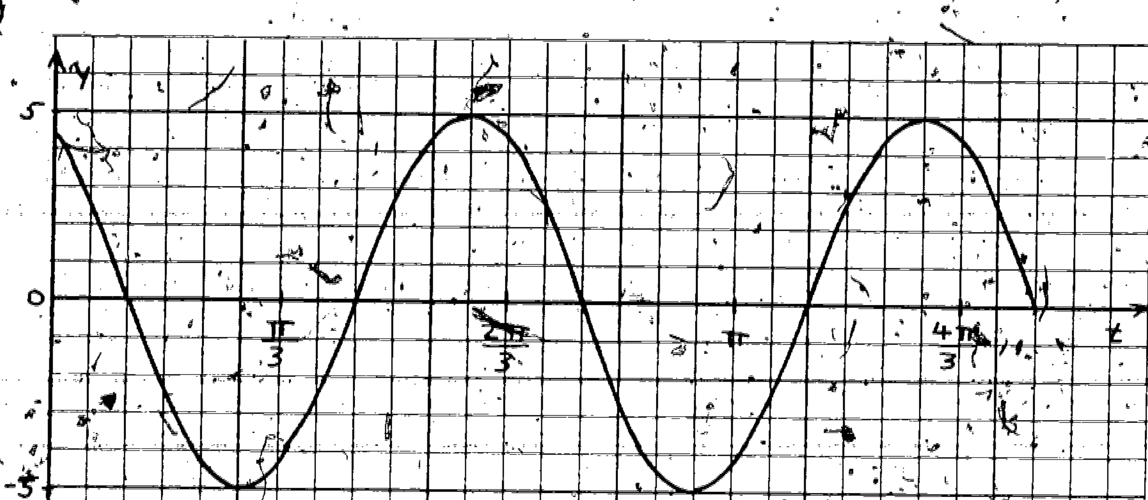
Amplitude = 2, Period = 6π, Phase lag = $9\pi/2$.

e)



Amplitude = 2, Period = π, Phase lag = $\pi/4$.

f)



Amplitude = 5, Period = $2\pi/3$, Phase lag = $11\pi/18$.

3. a) $y = A\cos(\omega t - \alpha) = A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha$

$y = 4\sin\pi t - 3\cos\pi t$

$A\sin\alpha = 4, \quad A\cos\alpha = -3$

$A^2(\sin^2\alpha + \cos^2\alpha) = 16 + 9$

$A^2 = 25$

$A = 5$

$\sin\alpha = \frac{4}{5}, \quad \cos\alpha = -\frac{3}{5},$

α is in II and $\alpha \approx \pi - 0.927 \approx 2.215$

Answer: $y = 5\cos(\pi t - 2.215)$

b) $y = -4\sin\pi t + 3\cos\pi t$

$A\sin\alpha = -4, \quad A\cos\alpha = 3, \quad A = 5.$

$\sin\alpha = -\frac{4}{5}, \quad \cos\alpha = \frac{3}{5},$

$\therefore$ α is in IV and $\alpha \approx 2\pi - 0.927 \approx 5.357$

Answer: $y = 5\cos(\pi t - 5.357)$

c) $y = -4\sin\pi t - 3\cos\pi t$

$A = 5, \quad \sin\alpha = -\frac{4}{5}, \quad \cos\alpha = -\frac{3}{5}.$

$\therefore$ α is in III and $\alpha \approx \pi + 0.927 \approx 4.069$

Answer: $y = 5\cos(\pi t - 4.069)$

d) $y = 3\sin\pi t + 4\cos\pi t$

$A = 5, \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = \frac{4}{5}$

α is in I and $\alpha \approx 0.644$

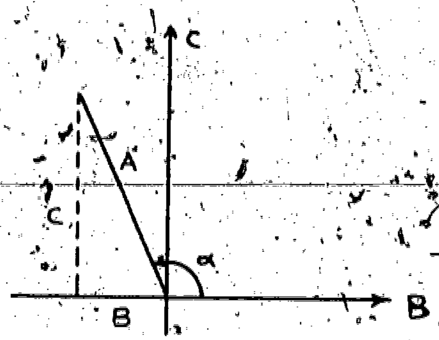
Answer: $y = 5\cos(\pi t - 0.644)$

e) $y = 3\sin\pi t - 4\cos\pi t$

$A = 5, \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = -\frac{4}{5},$

$\therefore$ α is in II and $\alpha \approx \pi - 0.644 \approx 2.498$

Answer: $y = 5\cos(\pi t - 2.498)$

4. 

$$A^2 = B^2 + C^2$$

$$\sin\alpha = \frac{C}{A}, \quad \cos\alpha = \frac{B}{A}.$$

Although the directions in this problem do not ask for the values of t at which the maxima and minima occur, they have been included in these solutions in case the question arises.

a) $A = 5$, $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\alpha \approx 0.644$.
Hence, $3\sin 2t + 4\cos 2t \approx 5\cos(2t - 0.644)$.
Maximum value of 5 occurs when $\cos(2t - 0.644) = 1$, or $2t - 0.644 = 0$, $t = 0.322$. Minimum value of -5 occurs when $\cos(2t - 0.644) = -1$, or $2t - 0.644 = \pi$, $t \approx 1.893$. The period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$.
Hence, maximum values occur at $t \approx 0.322 + n\pi$ and minimum values at $t \approx 1.893 + n\pi$.

b) $A = \sqrt{4+9} = \sqrt{13}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{-3}{\sqrt{13}}$,
$\alpha \approx \pi - 0.589 \approx 2.553$.
Hence, $2\sin 3t - 3\cos 3t \approx \sqrt{13}\cos(3t - 2.553)$.
The period $= \frac{2\pi}{3}$. Maximum values of $\sqrt{13}$ occur when $3t - 2.553 = 0 + 2n\pi$, $t \approx 0.851 + \frac{2n\pi}{3}$. Minimum values of $-\sqrt{13}$ occur when $3t - 2.553 = \pi + 2n\pi$, $t \approx 1.898 + \frac{2n\pi}{3}$.

4\. c) $A = \sqrt{1+1} = \sqrt{2}$, $\sin\alpha = -\frac{1}{\sqrt{2}}$, $\cos\alpha = \frac{1}{\sqrt{2}}$, $\alpha = \frac{7\pi}{4}$.

Hence, $-\sin(t/2) + \cos(t/2) = \sqrt{2}\cos(t/2 - 7\pi/4)$.

The period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. Maximum values of $\sqrt{2}$ occur when $\frac{t}{2} - \frac{7\pi}{4} = 0 + 2n\pi$, $t = \frac{7\pi}{2} + 4n\pi$. Minimum values of $-\sqrt{2}$ occur when $\frac{t}{2} - \frac{7\pi}{4} = \pi + 2n\pi$; $t = \frac{11\pi}{2} + 4n\pi$.

5\. $A\cos(\omega t - \alpha) + B\cos(\omega t - \beta)$

$= A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha + B\cos\omega t\cos\beta$
$+ B\sin\omega t\sin\beta$

$= (A\cos\alpha + B\cos\beta)\cos\omega t + (A\sin\alpha + B\sin\beta)\sin\omega t$

$= C\cos(\omega t - \gamma)$ when

$C = \sqrt{(A\sin\alpha + B\sin\beta)^2 + (A\cos\alpha + B\cos\beta)^2}$,

$\sin\gamma = \frac{A\sin\alpha + B\sin\beta}{C}$, and $\cos\gamma = \frac{A\cos\alpha + B\cos\beta}{C}$

Since A, B, α, and β are real numbers, it follows that C and γ are real numbers.

6\. a) From the solution in the text,

$$t = \frac{0.927}{\pi} \pm \frac{1}{3} \pm 2n.$$

So, $t \approx 0.295 \pm 0.333 \pm 2n$. The smallest positive value of t is $t \approx 0.295 + 0.333 = 0.628$.

b) $3\cos\pi t + 4\sin\pi t = 5$

$5\cos(\pi t - 0.927) = 5$

$\cos(\pi t - 0.927) = 1$

This is satisfied when the argument of the cosine is $0 \pm 2n\pi$. Therefore, $\pi t - 0.927 = \pm 2n\pi$, or $t \approx \frac{0.927}{\pi} \pm 2n \approx 0.295 \pm 2n$.

6\. b) The smallest positive value of t is $t \approx 0.295$.

c) $\sin 2t - \cos 2t = 1$

$$\sqrt{2}\cos\left(2t - \frac{3\pi}{4}\right) = 1$$

$$\cos\left(2t - \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

This is satisfied when the argument of the cosine is $\pm\frac{\pi}{4} \pm 2n\pi$. Therefore, $2t - \frac{3\pi}{4} = \pm\frac{\pi}{4} \pm 2n\pi$, or $t = \frac{3\pi}{8} \pm \frac{\pi}{8} \pm n\pi$. The smallest positive value of t is $t = \frac{\pi}{4}$.

d) $4 \cos \pi t - 3 \sin \pi t = 0$

$$5 \cos (\pi t - 5.640) = 0$$

$$\cos(\pi t - 5.640) = 0.$$

This is satisfied when the argument of the cosine is $\pm \frac{\pi}{2} \pm 2n\pi$. Therefore, $\pi t - 5.640 = \pm \frac{\pi}{2} \pm 2n\pi$, or $t \approx \frac{5.640}{\pi} \pm \frac{1}{2} \pm 2n, \approx 1.795 \pm 0.5 \pm 2n$. The smallest positive value of t is $t \approx 1.795 + 0.5 - 2$

$$\approx 0.295.$$

e) $4 \cos \pi t + 3 \sin \pi t = 1$

$$5 \cos(\pi t - 0.644) = 1$$

$$\cos(\pi t - 0.644) = 0.2$$

This is satisfied when the argument of the cosine is approximately $\pm 1.369 \pm 2n\pi$ (from Table I). Therefore, $\pi t - 0.644 \approx \pm 1.369 \pm 2n\pi$, or $t \approx \frac{0.644 \pm 1.369}{\pi} \pm 2n$. The smallest positive value of t is $t \approx \frac{0.644 - 1.369}{\pi} + 2, \approx -0.231 + 2, \approx 1.769$.

7. Given $y = B\cos(\mu t - \beta)$.

We may clearly assume that $0 \le \beta < 2\pi$.

1. If μ and B are positive, we set $\mu = \omega$, $B = A$, $\beta = \alpha$.
2. If μ is positive and B is negative, set $\mu = \omega$, $B = -A$.

   Then $y = A[-\cos(\omega t - \beta)] = A\cos(\omega t - \beta \pm \pi)$.

   If $0 \le \beta < \pi$, take $\alpha = \beta + \pi$.

   If $\pi \le \beta < 2\pi$, take $\alpha = \beta - \pi$.
3. If μ is negative, set $\mu = -\omega$.

   Then $y = B\cos(-\omega t - \beta) = B\cos(\omega t + \beta)$

   $= B\cos[\omega t - (2\pi - \beta)]$

   $= B\cos(\omega t - \beta')$.

   Proceed as in 1 and 2.

5-9 Identities. Tangent to $y = \cos x$.

Exercises 5-9a.

The identities dealt with here are somewhat more difficult than earlier ones. It may be necessary for you to work a few additional examples on the blackboard to help get the students started.

Solutions to Exercises 5-9a.

1. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$

$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$ (1)

$\cos(x - y) - \cos(x + y) = 2 \sin x \sin y$

$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$ (2)

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$ (3)

2. $\cos m\alpha \cdot \cos n\alpha = \frac{1}{2}[\cos(m + n)\alpha + \cos(m - n)\alpha]$

$\sin m\alpha \cdot \sin n\alpha = \frac{1}{2}[\cos(m - n)\alpha - \cos(m + n)\alpha]$

$\sin m\alpha \cdot \cos n\alpha = \frac{1}{2}[\sin(m + n)\alpha + \sin(m - n)\alpha]$

3. $\sin\alpha + \sin\beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$ (6)

Replace β by $-\beta$ and $\sin\beta$ by $-\sin\beta$. Then

$\sin\alpha - \sin\beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$.

4. In (5), let $\alpha = x$ and $\beta = \frac{\pi}{2} - x$. Then

$\cos\alpha - \cos\beta = \cos x - \sin x$

$= -2 \sin \frac{\pi}{4} \sin \left(\frac{2x - \pi/2}{2}\right)$

$= -\sqrt{2} \sin \left(x - \frac{\pi}{4}\right)$

$= \sqrt{2} \sin \left(\frac{\pi}{4} - x\right)$.

5. a) $\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6}$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$

5. b) $\cos \frac{5\pi}{12} = \cos(\frac{\pi}{4} + \frac{\pi}{6}) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$

$= (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) - (\frac{\sqrt{2}}{2})(\frac{1}{2}) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$\frac{1}{1} + \frac{\sqrt{3}}{\sqrt{3}}$

6.1 a)

6. e) $\tan^2 \frac{1}{2}\theta = \frac{\sin^2 \theta/2}{\cos^2 \theta/2} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{(1 - \cos\theta)(1 + \cos\theta)}{(1 + \cos\theta)^2}$

$$= \frac{\sin^2\theta}{(1 + \cos\theta)^2}$$

Hence $\tan \frac{1}{2}\theta = \pm \frac{\sin\theta}{1 + \cos\theta}$ [illegible]

[illegible]

Again [illegible]

[illegible]

6. 1) $\frac{2\tan\theta}{1+\tan^2} = \frac{\frac{2\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{2\sin\theta}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin^2\theta}$ [illegible]

2 sin θ cos θ [illegible]

2) $\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta} = \frac{(1+\cos\theta)^2 + \sin^2\theta}{\sin\theta(1+\cos\theta)}$

[illegible] sin θ (1 + cos θ)

$\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$

[illegible]

$\frac{\sin[illegible]}{\sin\alpha} - \frac{[illegible]}{\cos\alpha} = \frac{[illegible]}{\sin\alpha\cos\alpha}$ [illegible]

$\frac{\sin(2\alpha - \alpha)}{\sin\alpha\cos\alpha}$ [illegible] $\frac{\sin[illegible]}{\sin\alpha\cos[illegible]}$

[illegible]

7. b) $\cos^4 x = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{8}[3 + 4\cos 2x + \cos 4x]$

8. $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin^4 x = \frac{1}{4}($[illegible]$)$

$= \frac{1}{4}[1 - 2\cos 2x + \frac{1}{2}($[illegible]$)]$

$= \frac{1}{8}[3 - 4\cos 2x + \cos 4x]$

[illegible]

b) $\sin(x + y)$ [illegible] $(y + z)\cos$ [illegible]

$= (\sin x \cos y + \cos x \sin y)\cos z$

$+ (\sin y \cos z + \cos y \sin z)\cos$ [illegible]

[illegible]

9. e) Hence $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

f) $\sin 3x + \sin x = 2\sin 2x \cos x$ [by (6)]

Hence, $\sin x + \sin 2x + \sin 3x = \sin 2x[1 + 2\cos x]$

g) $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{1 + \dfrac{\sin x}{\cos x}}{1 - \dfrac{\sin x}{\cos x}} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

Hence

$$\left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \frac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{\cos^2 x - 2\sin x\cos x + \cos^2 x}$$

$$= \frac{1 + 2\sin x\cos x}{1 - 2\sin x\cos x}$$

$$= \frac{1 + \sin 2x}{1 - \sin 2x}$$

Slope of the tangent to the graph $y = \cos x$ at $x = a$.

We use the same ideas and the same symbolism here as we did in Chapter 2. If the ... of the ... as ... approaches ...

Solutions to Exercises 5-9b.

1. a) b) c) are special cases of d).

In d)

$$s(x) = \frac{\cos kx - \cos kh}{x - h}$$

$$= \frac{-2 \sin \frac{k}{2}(x + h) \sin \frac{k}{2}(x - h)}{x - h}$$

$$= -k \sin \frac{k}{2}(x + h) \frac{\sin \frac{k}{2}(x - h)}{\frac{k}{2}(x - h)}$$

As x approaches h, the bracket approaches 1 and s(x) approaches $-k \sin kh$.

The answers to a) b) c) are [illegible]

$-\frac{1}{2} \sin \frac{1}{2} h$ and [illegible]

2. y = sin x

[illegible] $\frac{\sin}{x - h}$ [illegible]

[illegible]

1. Hence $\frac{-\sin z}{2\cos z} \leq \frac{-z}{2}$ and $\frac{\sin x}{2\cos x} \leq \frac{x}{2}$.

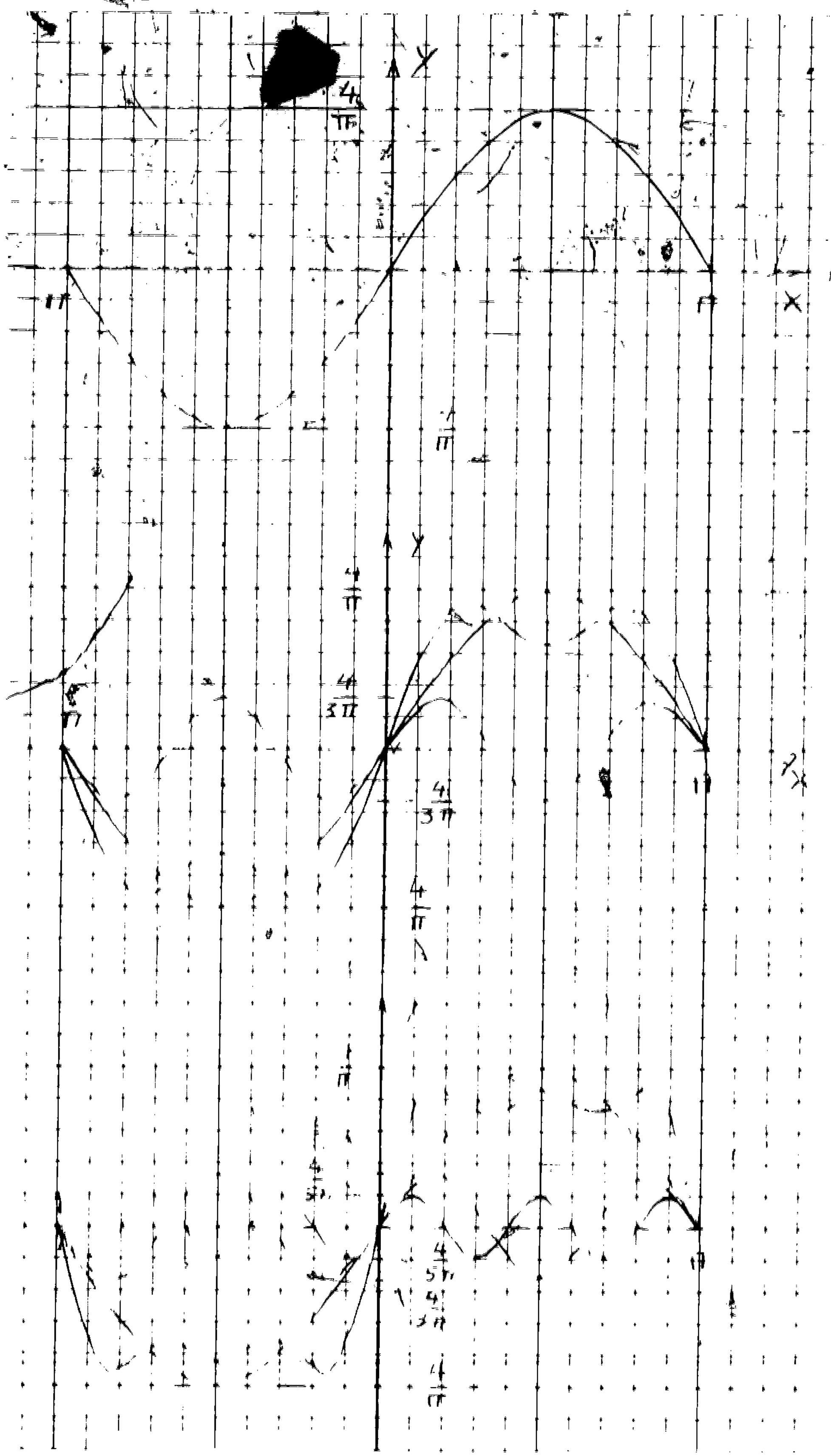
2. As in Exercise 1, let [illegible]. Then $\sin x$ [illegible] $\sin z$, $\cos x$ [illegible]. Since $\frac{\sin z}{z} \leq \frac{1}{\cos z}$ and $\frac{\sin z}{z} \geq \cos z$ and $\frac{\sin z}{z} = \frac{\sin x}{x}$ and $\cos z = \cos x$, $\frac{\sin x}{x}$ [illegible] $\cos x$ [illegible] $\frac{\sin x}{x}$ [illegible]

3. (a) $\frac{\sin \pi x}{x}$ [illegible] $\frac{\sin \pi x}{\pi x}$ approaches [illegible] approaches [illegible] Since $\frac{\sin t}{t}$ approaches [illegible] approaches [illegible]

(b) $\frac{\sin x^2}{x}$ [illegible] $x\left(\frac{\sin x^2}{x^2}\right)$ [illegible] as [illegible] approaches [illegible] limit [illegible]

(c) [illegible]

[illegible]


Y
4/π
π
π
X
Y
4/π
4/3π
π
4/3π
π
X
4/π
4/π
4/5π
π
4/5π
4/7π
4/π

2. a) 2π, $\frac{2\pi}{3}$, $\frac{2\pi}{5}$, ...

b) The cosine terms; also the terms $B_n \sin nx$, n even. (In our case, $a = 2\pi$)

c) The function being represented has the property that $f(-x) = -f(x)$ [odd function]. This property holds [illegible]

[illegible] Moreover, $f(x)$ has the [illegible] $f(x$ [illegible] $) = f(x)$. This property does not [illegible] sin [illegible] integral [illegible] sin [illegible]

[illegible]

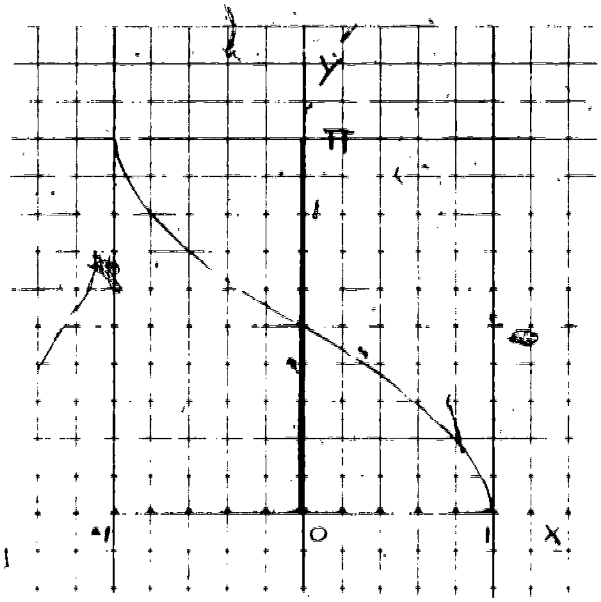
generally

[illegible]

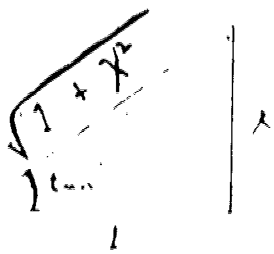
1. $y = \cos^{-1} x$

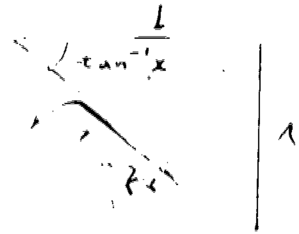
Domain $-1 \le x \le 1$

Range $0 \le y \le \pi$



5. $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1 + x^2}}$





$x > 0$

7. a) $\sin x + \cos x = 0$, $\tan x = -1$

$x = \frac{\pi}{4} \pm n\pi$

b) $4\cos^2 x - 1 = 0$, $\cos x = \pm\frac{1}{2}$

$x = \frac{\pi}{3} \pm 2n\pi$ or $\frac{2\pi}{3} \pm 2n\pi$

$x = 1.19$

d) 4 tan

$4\frac{\sin x}{\cos x}$

8. c) $2 \tan x + 3 = \frac{2}{\tan x}$ $(x \neq 0$ and $x \neq \pm n\pi)$

$2 \tan^2 x + 3 \tan x - 2 = 0$

$(2 \tan x - 1)(\tan x + 2) = 0$

$\tan x = \frac{1}{2}$ or $\tan x = -2$

$x = 0.46 \pm n\pi$ $x = \ldots 1.11 \pm n\pi$

d) $\cos 2x = 1 - \sin x$

$1 - 2 \sin^2 x = 1 - \sin x$

$-2 \sin^2 x = -\sin x$

$\sin x = 0, \sin x = \frac{1}{2}$

$x = \ldots \frac{\pi}{6} \ldots$

9. a) $2 \sin \ldots \frac{\pi}{4}$

$x = \ldots \frac{\pi}{8}$

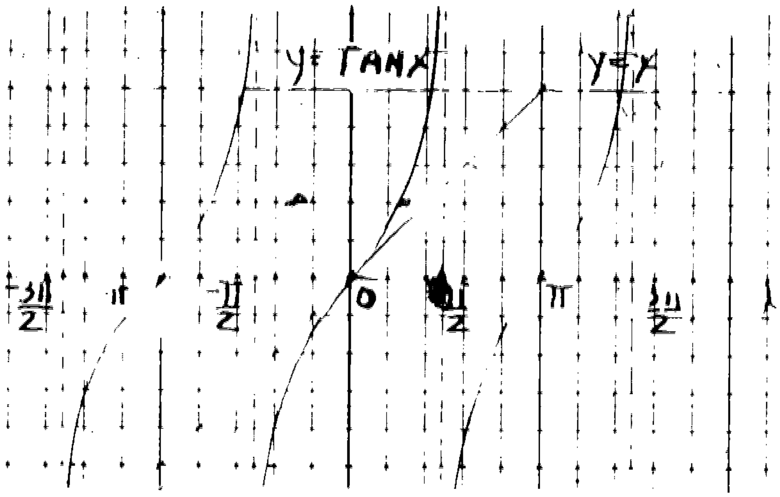
$\frac{1}{2}$

$\frac{\pi}{6}$

$\frac{1}{2}$

$x = \ldots 3$

10. a)



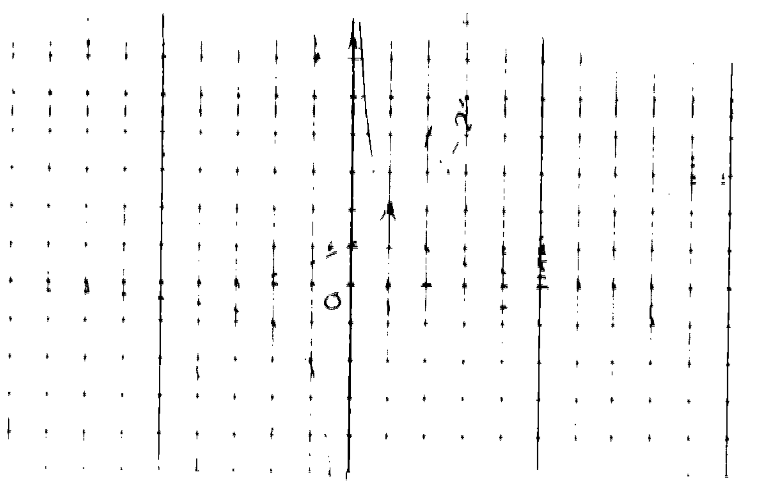
$x = \tan x$

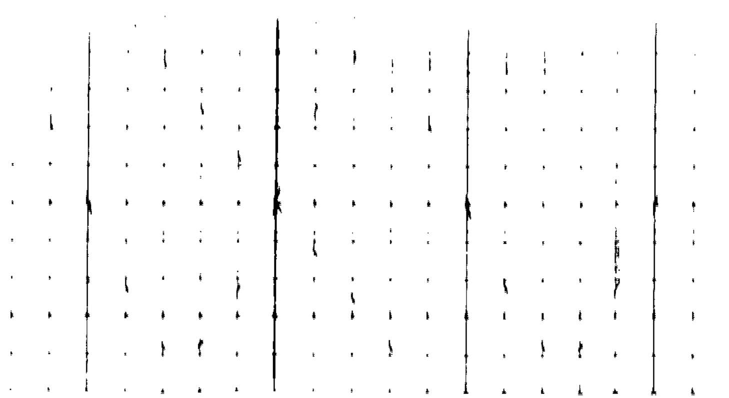
$x = 0, \pm 4.4$ [illegible]

and other soluti[illegible]

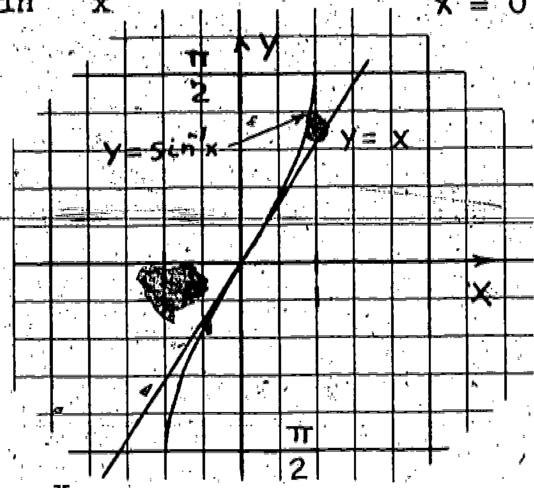
corresponding to

[illegible]

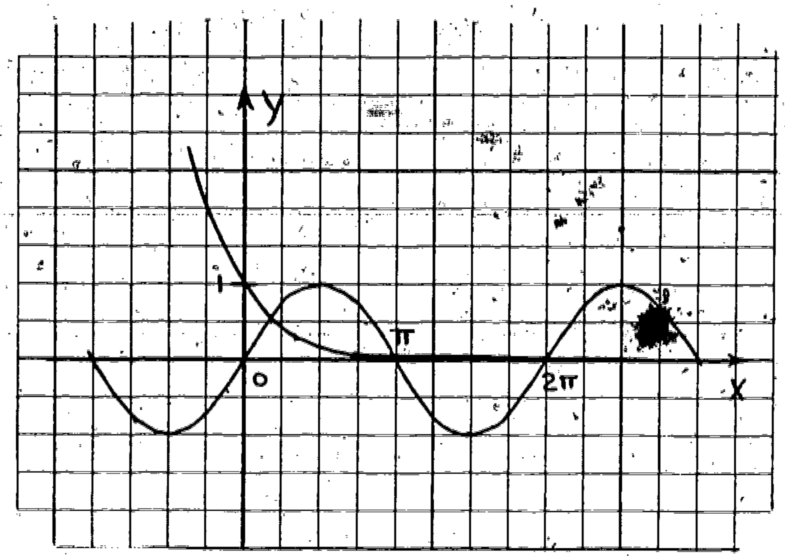




10. d) $x = \sin^{-1} x$ $x = 0$ is the only solution.



e) $\sin x = e^{-x}$ $x \approx .59,\ 3.12,$ and so on.



Miscellaneous Exercises

1. a) $|\sin(\pi + x)| = |-\sin x| = |\sin x|$. Periodic with period π.

b) Since $[x + 1] = [x] + 1$, we have

$$f(x + 1) = (x + 1) - [x + 1] = x + 1 - [x] - 1$$
$$= x - [x] = f(x).$$ periodic with period 1.

c) $y = x \sin x$. not periodic

d) $y = \sin^2 x$, $\sin^2(\pi + x) = [-\sin x]^2 = \sin^2 x$. period π.

e) $y = \sin x^2$ not periodic

f) $y = \dfrac{\sin x + 2 \cos x}{2 \sin x + \cos x}$. This is not defined if

$\sin x = \dfrac{-\cos x}{2}$ or if $\tan x = -\frac{1}{2}$ and therefore at

$x = \tan^{-1}(-\frac{1}{2}) \pm n\pi$. Defined for all other values of x.

$$f(x + \pi) = \frac{\sin(x + \pi) + 2 \cos(x + \pi)}{2 \sin(x + \pi) + \cos(x + \pi)} = \frac{-\sin x - 2 \cos x}{-2 \sin x - \cos x}$$

$$= \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} = f(x).$$ Therefore

periodic with period π if the domain is restricted.

g) $y = \sin x + |\sin x|$

$\sin x$ is periodic with period 2π.

$|\sin x|$ is periodic with period π

$\therefore$ The sum is periodic with period lcm of $2\pi + \pi$ or 2π

h) $y = \sin x + \sin(\sqrt{2}x)$

$\sin x$ is periodic with period 2π

$\sin \sqrt{2}x$ is periodic with period $\dfrac{2\pi}{\sqrt{2}}$

But there is no lcm of two incommenserable numbers therefore no period for the function.

2. The decimal expansion of $\frac{110}{909}$ is $.\dot{1}21\dot{0}$ range: $\{0, 1, 2\}$

$\therefore f(x + 4) = f(x)$ $\therefore$ periodic with period 4.

$f(97) = f(1 + 4 \cdot 24) = f(1) = 1.$

3. $f(x + 2) = f(x)$ $f(-x) = - f(x)$ $f(\frac{1}{2}) = 3$

a) $f(\frac{9}{2}) = f(\frac{1}{2} + 2 \cdot 2) = f(\frac{1}{2}) = 3$

b) $f(\frac{7}{2}) = f(- \frac{1}{2} + 4) = f(- \frac{1}{2}) = - f(\frac{1}{2}) = - 3$

c) $f(9) + f(-7) = f(1 + 4 \cdot 2) + f(-1 + 4 \cdot 2) = f(1) + f(-1)$

$= f(1) - f(1) = 0$

4. a) $\frac{22}{7}$ radians $\rightarrow (\frac{22}{7} \cdot \frac{180}{\pi})^\circ \approx 180^\circ$

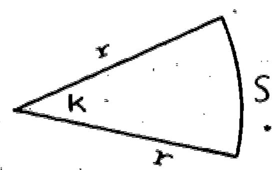
b) $\frac{2}{\pi}$ radians $\rightarrow (\frac{2}{\pi} \cdot \frac{180}{\pi})^\circ = (\frac{360}{\pi^2})^\circ$

5. a) $87^\circ \rightarrow \frac{87\pi}{180}$ radians

b) $(\frac{2}{\pi})^\circ \rightarrow \frac{2}{\pi} \cdot \frac{\pi}{180} = \frac{1}{90}$ radians

c) $(\frac{\pi}{2})^\circ \rightarrow \frac{\pi}{2} \cdot \frac{\pi}{180} = \frac{\pi^2}{360}$ radians

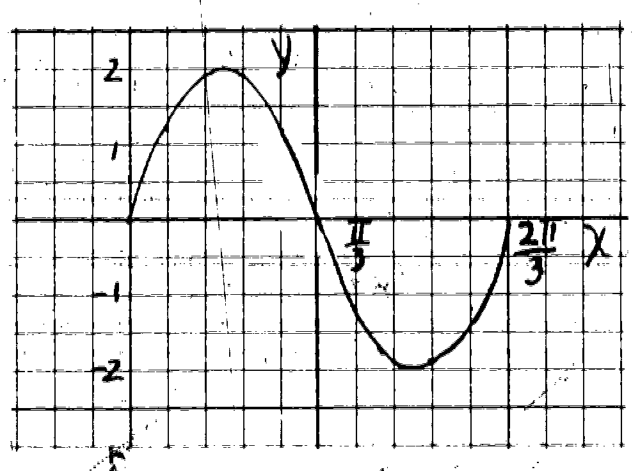
6. $S = kr$



$c = 2r + kr$, or $r = \frac{c}{2 + k}$

Area $= \frac{k}{2} \cdot r^2 = \frac{k}{2} \cdot \frac{c^2}{(2 + k)^2} = \frac{kc^2}{2(k + 2)^2}$

7. a) $y = 2 \sin 3x$: period $\frac{2\pi}{3}$, amplitude 2 range $[-2, 2]$.

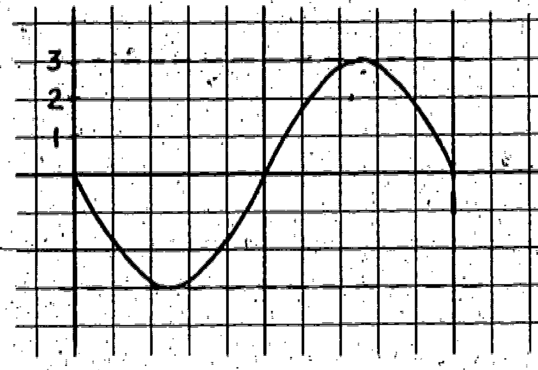


7. b) $y = -3 \sin 2\pi x$

period $\frac{2\pi}{2\pi} = 1$

Amplitude $+3$

Range $[-3, 3]$

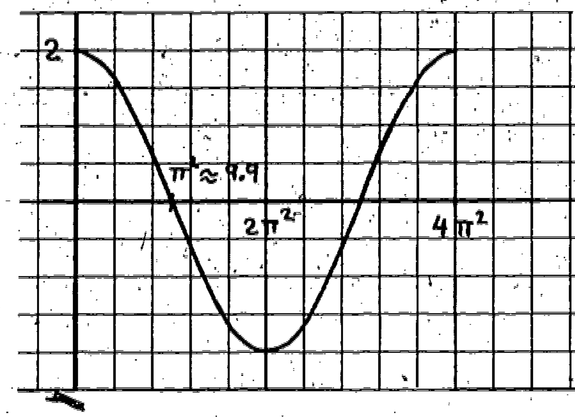


c) $y = 2 \cos \frac{x}{2\pi}$

period $\frac{2\pi}{\frac{1}{2\pi}} = 4\pi^2$

Amplitude 2

Range $[-2, 2]$



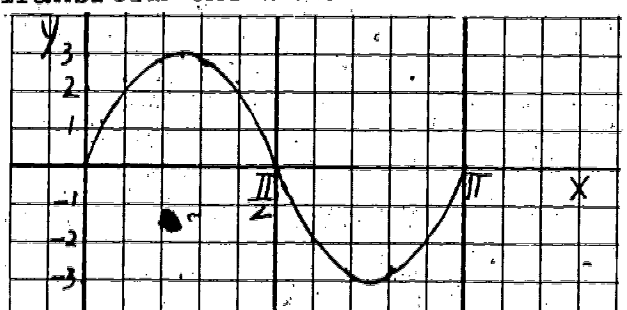
d) $y = 6 \sin x \cos x$. Transform the function first.

$y = 3 \sin 2x$.

period $\frac{2\pi}{2} = \pi$

Amplitude 3

Range $[-3, 3]$



e) $y = \sqrt{3} \sin 2x + \cos 2x$. Transform the function first

$y = [2\ \frac{\sqrt{3}}{2} \sin 2x + \frac{1}{2} \cos 2x]$ but $\sin \frac{\pi}{6} = \frac{1}{2}$ $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

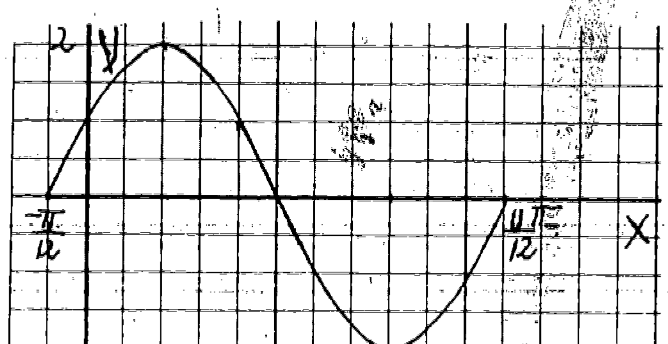
$y = 2(\sin 2x \cos \frac{\pi}{6} + \cos 2x \sin \frac{\pi}{6})$

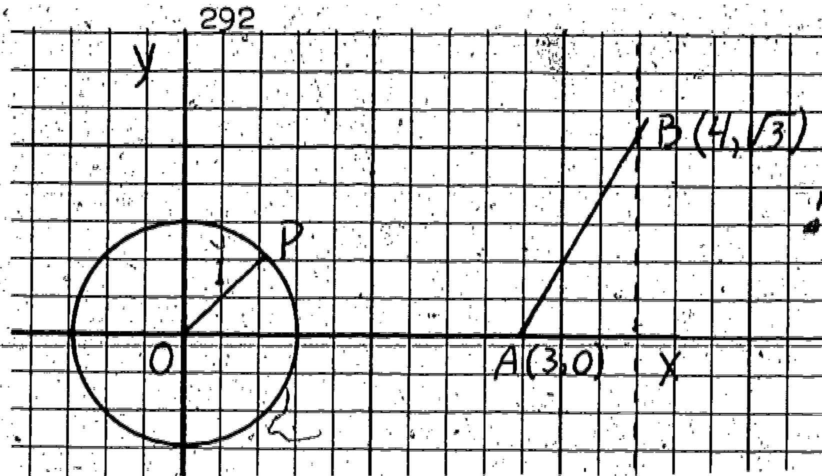
$y = 2 \sin(2x + \frac{\pi}{6})$

period $= \frac{2\pi}{2} = \pi$

Amplitude $= 2$

Range $[-2, 2]$





8. $OP = \frac{1}{2}U + \frac{\sqrt{3}}{2}V$

$x = \frac{\pi}{3}$

$OP = R_{\pi/3}(U)$

9. $f : x \to 3\cos(2\pi x + \frac{\pi}{2}) = -3\sin 2\pi x$

$= 3\sin(2\pi x + \pi)$

We can use $A = 3,\ B = 2\pi,\ C = \pi$

or $A = -3,\ B = 2\pi,\ C = 0$

10. $Q = Q_0 \sin[\frac{t}{\sqrt{Lc}} + \frac{\pi}{2}]$ $\quad L = 0.4$, $\quad c = 10^{-5}$

$Q = Q_0 \sin(500t + \frac{\pi}{2})$ $\qquad \frac{t}{\sqrt{Lc}} = \frac{t}{\sqrt{(4)(10^{-6})}} = \frac{t}{2 \cdot 10^{-3}} = 500t$

$= Q_0 \cos 500t$

a) period $= \frac{2\pi}{500}$, frequency $= \frac{250}{\pi}$

b) $Q = 0$ if $\cos 500t = 0$

This is zero for the first time at $t = \frac{1}{4} \cdot \frac{\pi}{250} = \frac{\pi}{1000}$.

c) $Q = .5Q_0$ if $\cos 500t = .5 = \frac{1}{2}$; $500t = \frac{\pi}{3}$ or $t = \frac{\pi}{1500}$

d) $Q = .5Q_0$ for second time $\quad 500t = \frac{5\pi}{3}$ or $t = \frac{\pi}{300}$

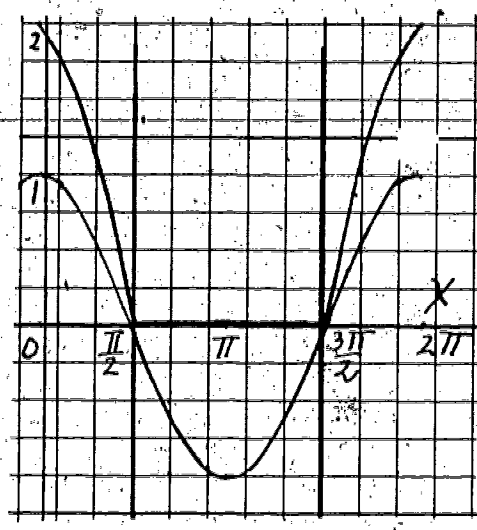
11. $$\tan\frac{3x}{2} = \frac{\cos x - \cos 2x}{\sin 2x - \sin x}$$

$$= -\frac{\cos 2x - \cos x}{\sin 2x - \sin x}$$

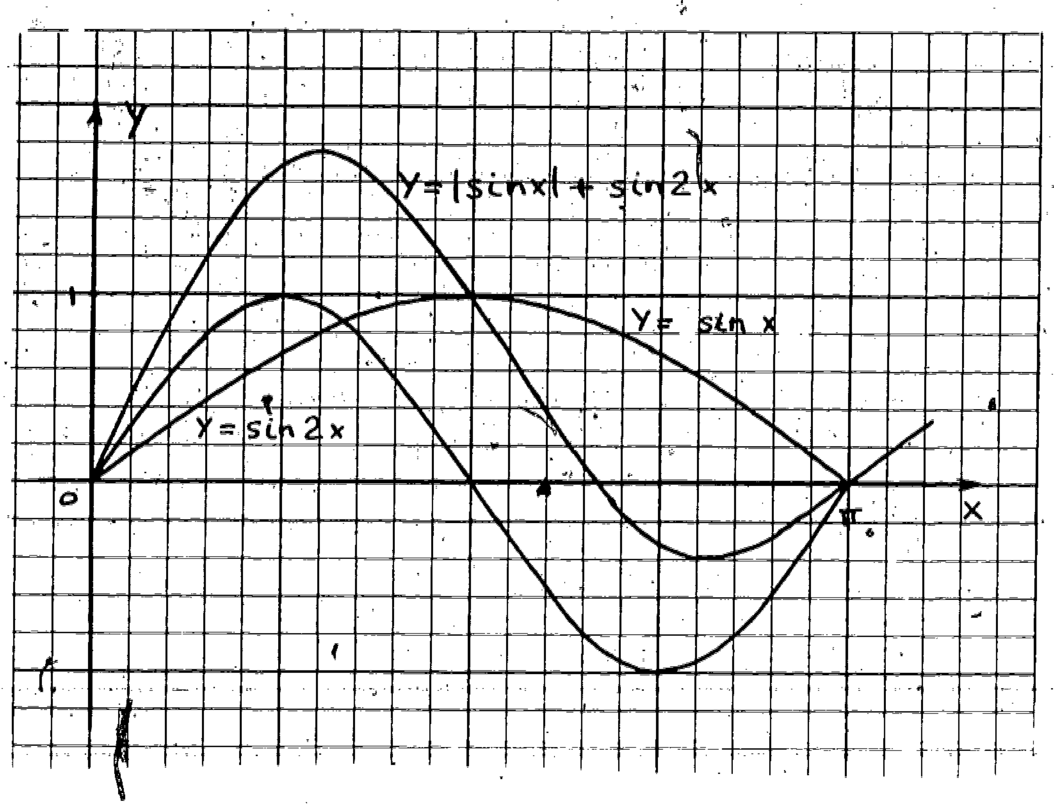
$$= -\frac{-2\sin\frac{1}{2}(2x + x)\sin\frac{1}{2}(2x - x)}{2\cos\frac{1}{2}(2x + x)\sin\frac{1}{2}(2x - x)}$$

$$= \frac{\sin\frac{3x}{2}}{\cos\frac{3x}{2}} = \tan\frac{3x}{2}.$$

12. Sketch graph of: a) $y = \cos x + |\cos x|$



b) $y = |\sin x| + \sin 2x.$



13. Prove : a) $|\sin x + \cos x| \leq \sqrt{2}$

$|\sin x + \cos x| = |\sqrt{2}\,[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x]|$

$= \sqrt{2}\,|\sin(x + \frac{\pi}{4})|$

and $|\sin\theta| \leq 1$. Q.E.D.

b) $|\sqrt{3}\sin x + \cos x| = |2\,[\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x]|$

$= |2|\,|\sin(x + \frac{\pi}{6})|$

≤ 2 since $|\sin\theta| \leq 1$.

14. Find $f(x + y)$ in terms of $f(x)$ and $f(y)$ only, given that: (Hint: Find x in terms of $f(x)$ and y in te s of $f(y)$)

$x \to \frac{x+2}{x+1}$ Answer: $\frac{f(x) + f(y) - 2}{2f(x) + 2f(y) - f(x)f(y) - 3}$

$x \to 1 + \frac{2}{x}$ Answer: $\frac{f(x)f(y) - 1}{f(x) + f(y) - 2}$

$x \to \frac{1}{x}$ Answer: $\frac{f(x)f(y)}{f(x) + f(y)}$

$x \quad 2^x$ Answer: $f(x)f(y)$

$x \to \log x$ Answer: $\log(10^{f(x)} + 10^{f(y)})$

15. a) $\sin x > 0$. Implies x in I or II Quadrant. Since $\sin x \cos x < 0$ and $\sin x > 0$, $\cos x < 0$ and x in II or III. Hence x is in II.

b) $\cos x < 0$ x in II or III, $\sin x \cos x > 0$ which first means $\sin x < 0$ or x in III or IV. Hence x is in III.

c) $x = 100$ $100 \approx (15.9)2\pi$ Therefore x is in IV.

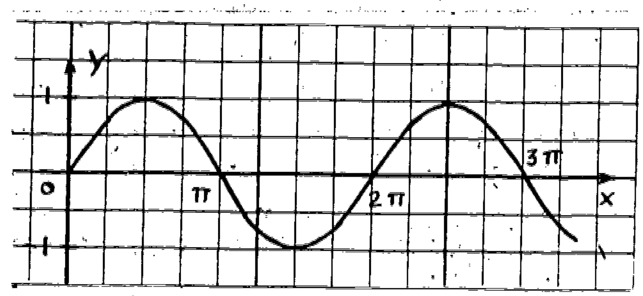
16. $\sin(\cos x) = \cos(\sin x)$ has no solution.

Proof: There could be no solution in quadrants II or III since the left member would be negative and the right member would be positive (if $\theta = \sin x$ we have $-1 \leq \theta \leq 1$ so that θ is in either quadrant I or IV.)

If there is a IV quadrant solution we may set $x = -z$. Then $\sin(\cos z) = \cos(-\sin z) = \cos(\sin z)$. Hence there is no IV quadrant solution unless there is a I quadrant solution. Then, if there is a I quadrant solution x, $\sin(\cos x) = \sin(\frac{\pi}{2} - \sin x)$ and $\sin x + \cos x = \frac{\pi}{2}$, or $\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = \frac{\pi}{2\sqrt{2}}$, $\sin(x + \frac{\pi}{4}) = \frac{\pi}{2\sqrt{2}}$. But this is impossible since $\frac{\pi}{2\sqrt{2}} \approx \frac{3.14}{2.28} > 1$.

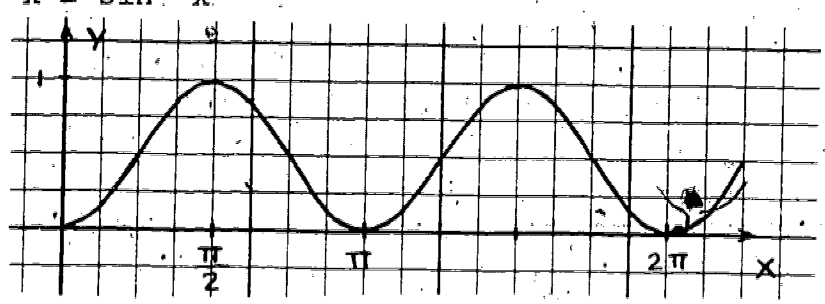
17. a) $y = \sin x$

[Also (h)]



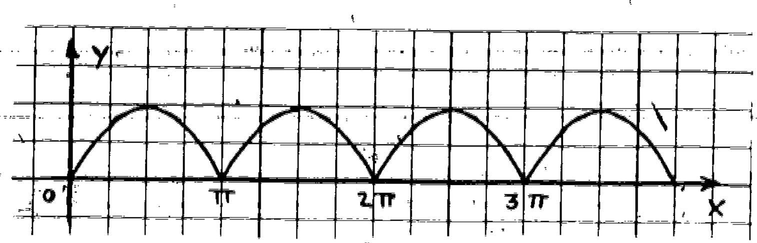
b) $y = 1 - \cos^2 x = \sin^2 x$

[Also (f)]

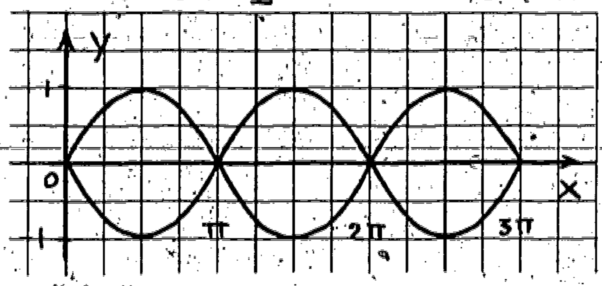


c) $y = |\sin x|$

[Also (k)].



d) $y^2 = \sin^2 x$, $y = \pm \sin x$ [Also (e), (g) (ℓ)]



e) Same as (d)

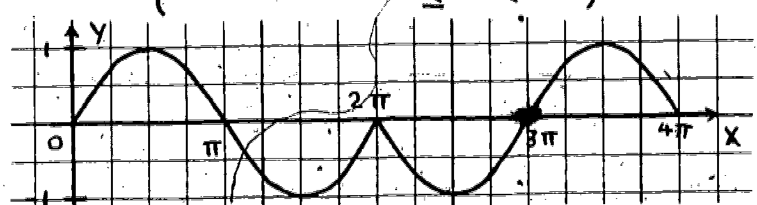
f) $y = \dfrac{1 - \cos 2x}{2} = \dfrac{1 - (1 - 2\sin^2 x)}{2} = \sin^2 x$

g) $y^2 = \sin^2 x$ [see (d)]

h) $y = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$ [See (1)]

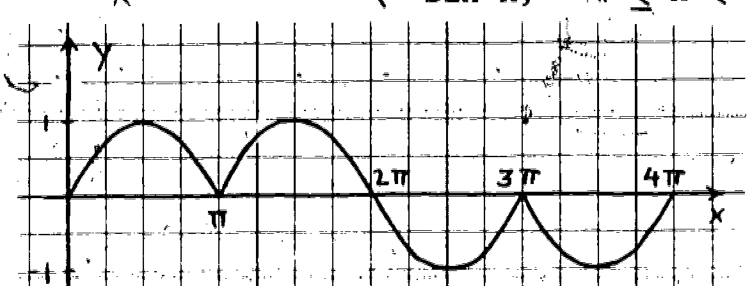
i) $y = 2 \left|\sin \frac{x}{2}\right| \cos \frac{x}{2} = \qquad = \sin x \dfrac{\left|\sin \frac{x}{2}\right|}{\sin \frac{x}{2}}$

$$= \begin{cases} \sin x & 0 \le x < 2\pi \\ -\sin x & 2\pi \le x < 4\pi \end{cases}$$



j) $y = 2 \sin \frac{x}{2} \left|\cos \frac{x}{2}\right| = \sin x \dfrac{|\cos x/2|}{\cos x/2}$

$$= \begin{cases} \sin x & 0 \le x < \pi \\ & 3\pi \le x < 4\pi \\ -\sin x, & \pi \le x < 3\pi \end{cases}$$



k) $y = 2 \left|\sin \frac{x}{2} \cos \frac{x}{2}\right| = |\sin x|$ [See (c)]

ℓ) $y^2 = 4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} = \sin^2 x$ [See (d)]

Illustrative Test Questions

1. Determine whether each of the following functions is periodic and, if so, find the fundamental period:

   (a) $y = |\cos 2x|$

   (b) $y = \sin 3x \cos 3x$

2. Given that $f: x \longrightarrow f(x)$ is periodic with fundamental period $\frac{1}{2}$ and given that $f(\frac{1}{4}) = 2$, $f(2) = 5$, and $f(\frac{11}{8}) = 3$, find

   (a) $f(0)$

   (b) $f(-\frac{5}{8})$

   (c) $f(\frac{3}{4})$

3. Sketch two complete periods of the graph of $y = 2 \sin 3x$.

4. Change from radians to degrees:

   (a) $\frac{7\pi}{12}$

   (b) $\frac{2}{15}$

5. Change from degrees to radians:

   (a) 165°

   (b) 2°

6. What is the radius of a circle in which a sector of area 6 has a perimeter 10?

   (two solutions)

7. Sketch the graph of $y = \sin x - \sqrt{3} \cos x$ over a complete period, indicating both the fundamental period and the amplitude.

8. Express $\sin (x + 2y)$ in terms of $\sin x$, $\sin y$, $\cos x$, $\cos y$.

9. Express the following in the form $\pm \sin x$ or $\pm \cos x$:

(a) $\sin (x + \frac{3\pi}{2})$

(b) $\cos (\frac{5\pi}{2} - x)$

(c) $\sin (- 3\pi - x)$

(d) $\cos (x + 5\pi)$

10. Show that

$$(\sin x + \sin 2x)(\sin x)(1 - 2 \cos x) = (\cos x + \cos 2x)(\cos 2x - \cos x)$$

holds for all real values of x.

11. Given $\sin 27^\circ = 0.4540$ and $\sin 28^\circ = 0.4695$, interpolate to find

(a) $\sin 27.4^\circ$

(b) the angle between 27° and 28° whose sine is 0.4664.

12. Given the function $x \to - 3 \sin (2x + \frac{\pi}{3})$, find the points on the graph with smallest positive x for which

(a) the function has the value zero

(b) the function has a maximum value

(c) the function has a minimum value.

*13. If a, b, c are constants, find A and B such that $\sin (x + c) = A \sin(x + a) + B \sin (x + b)$ holds for all values of x. (You may assume that $\sin (a - b) \neq 0$.)

14. Find (a) the limit of $\frac{\sin 3\pi x}{x}$ as x approahces zero, and (b) the slope of $y = \cos 3\pi x$ at $x = \frac{2}{9}$.

15. Evaluate (a) $\cos(\sin^{-1}(-\frac{1}{2}))$

(b) $\sin (\cos^{-1}(-\frac{1}{2}))$

*16. Consider the function $f : x \to \cos(\sin^{-1} x)$; $-1 \leq x \leq 1$.

(a) Find an algebraic expression for $f(x)$.

(b) What is the range of f ?

(c) Does f have an inverse ?

17. Find the values of x in the interval $0 \leq x < 2\pi$ which satisfy

(a) $\sin(3x + \frac{\pi}{2}) = \cos(\frac{\pi}{3} - 2x)$

(b) $\sin 2x - \cos 2x = \frac{\sqrt{6}}{2}$

Answers to Illustrative Test Questions

1. (a) $\cos 2(x + \frac{\pi}{2}) = \cos (2x + \pi) = -\cos 2x$

   Hence, $|\cos 2(x + \frac{\pi}{2})| = |\cos 2x|$ and the period is $\frac{\pi}{2}$

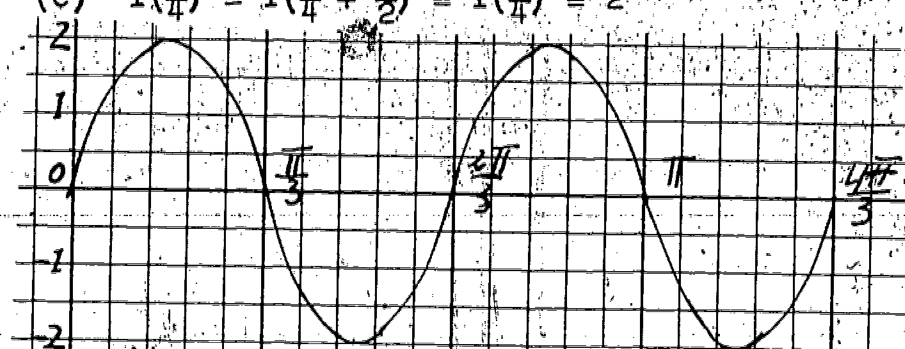
   (b) $\sin 3x \cos 3x = \frac{1}{2} \sin 6x = \frac{1}{2} \sin(6x + 2\pi)$

   $= \frac{1}{2} \sin 6(x + \frac{\pi}{3})$ and the period is $\frac{\pi}{3}$.

2. (a) $f(0) = f(2 - 4 \cdot \frac{1}{2}) = f(2) = 5$

   (b) $f(-\frac{5}{8}) = f(\frac{11}{8} - 4 \cdot \frac{1}{2}) = f(\frac{11}{8}) = 3$

   (c) $f(\frac{3}{4}) = f(\frac{1}{4} + \frac{1}{2}) = f(\frac{1}{4}) = 2$

3. 

4. (a) $\frac{7\pi}{12} \cdot \frac{180^\circ}{\pi} = 105^\circ$

   (b) $\frac{2}{15} \cdot \frac{180^\circ}{\pi} = \frac{24^\circ}{\pi}$

5. (a) $165^\circ \cdot \frac{\pi}{180^\circ} = \frac{11\pi}{12}$

   (b) $2^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{90}$

6. If the radius is r and the arc s, then

   $\frac{1}{2} r^2 s = 6$; $2r + s = 10$; $\frac{1}{2} r^2 (10 - 2r) = 6$;

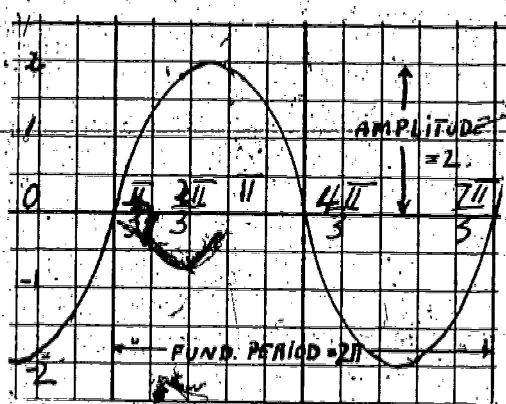
   $5r^2 - r^3 = 6$; $r^3 - 5r^2 + 6 = 0$; $(r + 1)(r^2 - 6r + 6) = 0$

   $r = -1$ or $3 \pm \sqrt{3}$. The negative result is meaningless.

   If $r = 3 + \sqrt{3} \approx 4.73$, then $s = 4 - 2\sqrt{3} \approx 0.54$.

   If $r = 3 - \sqrt{3} \approx 1.27$, then $s = 4 + 2\sqrt{3} \approx 7.46$.

7. Note that $y = \sin x - \sqrt{3}\cos x = 2\sin(x - \frac{\pi}{3})$ for all x.



8. $\sin(x + 2y) = \sin x \cos 2y + \cos x \sin 2y$

$= \sin x\,(\cos^2 y - \sin^2 y) + \cos x \cdot 2 \sin x \cos y$

$= \sin x \cos^2 y - \sin x \sin^2 y + 2 \sin x \cos x \cos y$

or $\sin x - 2 \sin x \sin^2 y + 2 \sin x \cos x \cos y$

9. (a) $-\cos x$

(b) $\sin x$

(c) $\sin x$

(d) $-\cos x$

10. $(\sin x + \sin 2x)(\sin x)(1 - 2\cos x)$

$= (\cos x + \cos 2x)(\cos 2x - \cos x)$

$(\sin x + \sin 2x)(\sin x - 2 \sin x \cos x)$

$(\sin x + \sin 2x)(\sin x - \sin 2x)$

$\sin^2 x - \sin^2 2x$

$1 - \cos^2 x - (1 - \cos^2 2x)$

$\cos^2 2x - \cos^2 x$

$(\cos 2x + \cos x)(\cos 2x - \cos x)$

11. (a) $\sin 28^\circ = 0.4695$

$\sin 27^\circ = \underline{0.4540}$

0.0155

$\underline{.4}$

$.00620$

$\underline{.4540}$

$\sin 27.4^\circ = .4602$

(b) .4064

$\underline{.4540}$

.0124

.0155

12. (a) $\ldots \sin(2x \ldots \frac{\pi}{3}) \ldots$

$\frac{\pi}{3}$

$\frac{\pi}{3}$ $\frac{\pi}{3}$ $\frac{\pi}{3}$

$\frac{\pi}{6}$ $\frac{\pi}{3}$ $\frac{2\pi}{3}$

3

3

$\frac{\pi}{3}$ $\frac{\pi}{2}$

$\frac{\pi}{12}$ $\frac{\pi}{12}$

$\frac{\pi}{2}$

$\frac{\pi}{6}$

12. (c) $x = \frac{\pi}{12}, \; -\frac{11\pi}{12}, \; \ldots$

Answer: $\frac{\pi}{12}$

13. If $x = -a$, we have

$$\sin(c - a) = A \sin 0 + B \sin(b - a)$$

$$B = \frac{\sin(c - a)}{\sin(b - a)} = \frac{\sin(a - c)}{\sin(a - b)}$$

If $x = -b$, we have

$$\sin(c - b) = A \sin(a - b) + B \sin 0$$

$$A = \frac{\sin(c - b)}{\sin(a - b)} = \frac{\sin(b - c)}{\sin(b - a)} = \frac{\sin(b - c)}{\sin(a - b)}$$

14. (a) $\frac{\sin 3\pi x}{x} = 3\pi \frac{\sin 3\pi x}{3\pi x}$ [illegible] the limit 3π

(b) If $f(x) = \cos 3\pi x$, then $f'(x) = -3\pi \sin 3\pi x$

and $f'(\frac{2}{9}) = -3\pi \sin \frac{2\pi}{3} = -3\pi(\frac{\sqrt{3}}{2}) = -\frac{3\pi\sqrt{3}}{2}$

15. (a) [illegible] $(-\frac{1}{2})$, [illegible] $(-\frac{\pi}{6})$ [illegible] $\frac{1}{2}$

(b) [illegible] $(-\frac{1}{2})$) [illegible] $\frac{\sqrt{3}}{2}$

16. (a) $f(x)$ [illegible]

If we set y [illegible]

$f(x)$ [illegible] y then [illegible]

and [illegible]

$\frac{\pi}{2}$ [illegible] $y \le \frac{\pi}{2}$ [illegible]

[illegible]

[illegible] $f(x)$ [illegible]

( ) [illegible]

then [illegible]

an inverse

17. (a) Since $\sin(y + \frac{\pi}{2}) = \cos y$, we can transform the equation to $\cos 3x = \cos(\frac{\pi}{3} - 2x)$

$$\cos 3x - \cos(\frac{\pi}{3} - 2x) = 0$$

Using (5) on Page 370, we obtain

$$-2 \sin(\frac{x}{2} + \frac{\pi}{6}) \sin(\frac{5x}{2} - \frac{\pi}{6}) = 0,$$ hence

$\sin(\frac{x}{2} + \frac{\pi}{6}) = 0$ or $\sin(\frac{5x}{2} - \frac{\pi}{6}) = 0$

$\frac{x}{2} + \frac{\pi}{6} = 0, \pi, \ldots$ $\frac{5x}{2} - \frac{\pi}{6} = 0, \ldots$

$\frac{x}{2} = -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$ $\frac{5x}{2} = \frac{\pi}{6}, \ldots$

$x = -\frac{\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{3}$ $\frac{\pi}{15}, \ldots$

[illegible]

Appendices

*2.7 Mathematical Induction

Both the teacher and the students can have a great deal of fun with this topic. The section should not be attempted with a below-average class. For an average class it is probably wise to eliminate the second principle of mathematical induction (and, of course, all exercises which depend upon it) as well as the Examples 10 and 11 of false proofs by induction. The false propositions "proved" in these examples are deliberately outrageous on first sight so that even the poorest student will be aware that there is a flaw in the logic whether or not he can find it. The flaw in Example 10 is, of course, that the initial step fails: 1 is not an even number. In Example 11, it is the sequential step that fails, [illegible]

[illegible]

Solutions to Exercises (2-7) on Mathematical Induction.

1. (First principle).

Initial Step: For $n = 1$, $\frac{1}{2}n(n+1) = 1$.

Sequential Step: If the result is true for k, that is, if

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1),$$

then

$$\begin{aligned}(1 + 2 + 3 + \cdots + k) + (k+1) &= \frac{1}{2}k(k+1) + k + 1\\ &= (k+1)\left(\frac{1}{2}k + 1\right)\\ &= \frac{1}{2}(k+1)(k+2)\end{aligned}$$

2a. (First principle).

Initial Step: For [illegible]

[illegible]

Sequential Step: [illegible] If the result is true for k [illegible]

[illegible] $(a + k\ldots)$

[illegible]

2b. (First Principle)

<u>Initial Step</u>: For n = 1,

$$\frac{a(r^n - 1)}{r - 1} = a$$

provided $r \neq 1$. (point out to the class that the sum formula is not valid when $r = 1$.)

<u>Sequential Step</u>: Denote the s[illegible] the series to the first n terms by S_n. If the result is true[illegible] $n = k$, then

$$S_{k+1} = S_k + ar^k = \frac{a(r^k - 1)}{r-1} + ar^k$$

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

Initial Step:

4. (First Principle)

<u>Initial Step</u>: $2 \cdot 1 = 2 = 2^1$.

<u>Sequential Step</u>: Let us assume the truth of the assertion for $n = k$, that is

$$2^k \geq 2k.$$

On multiplying by 2 we have

$$2 \cdot 2^k = 2^{k+1} \geq 2(2k) = 2k + 2k$$

On the other hand k is a natural number and thus for k ≥ 1 it follows that $2k \geq 2$ and $2k + 2k \geq 2k + 2$. Consequently,

$$2^{k+1} \geq 2k + 2k \geq 2k + 2 = 2(k+1).$$

5. (First principle)

<u>Initial Step</u>:

$$(1 + p)^n$$

<u>Sequential Step</u>:

$n = k$, that is

$$(1 + p)(1 + kp)$$

$$(1 + p)^{k+1}$$

6. (First Principle)

Initial Step: For n = 1 the relation

$$1 = 1 + (n - 1)2^n$$

is plainly satisfied.

Sequential Step:

Let S_n denote the sum of the series to n terms. If the theorem is true for n = k then

$$S_{k+1} = S_k + (k + 1)2^k$$
$$= 1 + (k - 1)2^k + (k + 1)2^k$$
$$= 1 + [(k - 1) + (k + 1)]2^k$$
$$= 1 + (2k)2^k$$
$$= 1 + k2^{k+1}.$$

7a [illegible]

The student may [illegible] the question arises you [illegible] point [illegible] composite factors and that [illegible] composite factors and [illegible] get [illegible]

Initial Step

For [illegible]

[illegible]

Suppose the [illegible]

[illegible]

a) [illegible]

b) k + 1 [illegible]

other words

$$k + 1 = ab$$

where a and b are natural numbers. From $a \neq 1$ we have $b \neq k + 1$, and from $a \neq k + 1$ we have $b \neq 1$. It follows for both that $1 < a,\ b \leq k$. Clearly, then a, b are either prime or factorable into primes and the desired factorization of $k + 1$ is obtained by forming their product

7b. (Second Principle)

<u>Initial Step</u>: For $n = 1$, the number U_1 is defined [illegible]

<u>Sequential Step</u>: Suppose that for all natural numbers less than or equal to k that the assertion is true. We know for U_{k+1} that there exist natural numbers p, q and [illegible] and

$$U_{k+1} = U_p + U_q.$$

Since p is a natural number [illegible] and it follows that [illegible] Similarly, from [illegible] it follows that [illegible] and q are [illegible] than or equal [illegible] we have [illegible]

[illegible]

[illegible]

[illegible] U [illegible]

point out that what really has been proved is that [illegible] numbers U with the properties [illegible] the numbers we [illegible] the properties [illegible]

8. (First Principle)

The student should be expected to compile a table for a few values of the sum S_n to n terms:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{2\cdot 3} = \frac{4}{2\cdot 3} = \frac{2}{3}$$

$$S_3 = \frac{2}{3} + \frac{1}{3\cdot 4} = \frac{9}{3\cdot 4} = \frac{3}{4}$$

$$S_4 = \frac{3}{4} + \frac{1}{4\cdot 5} = \frac{16}{4\cdot 5} = \frac{4}{5}$$

$$\cdots$$

At this point he will be well on his way to finding the correct hypothesis and to establishing his proof by induction.

Hypothesis. For all natural numbers n the sum S_n to n terms of the given series satisfies

[illegible]

[illegible]

[illegible]

$$[illegible] (k + 1)(k + 2)$$

$$[illegible] (k + 1)(k + 2)$$

$$(k + 1)[illegible]$$

$$(k + 1)(k + [illegible])$$

$$[illegible]$$

$$k + 2$$

After the student has worked through this problem systematically, you may wish to point out a quick proof using the fact that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\text{, so that}$$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

The only trouble with this is that the student may fail to realize that a proof by mathematical induction is still necessary.

9. (First Principle)

There are many possible attacks on this problem. Here we observe on tabulating a few values of the sum S_n to n terms:

$S_1 = 1$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Proof:

<u>Initial Step</u>:

$S_1 = 1$ is satisfied

<u>Sequential Step</u>: If the hypothesis is true for $n = k$ then

$$S_{k+1} = S_k + (k+1)^3$$

$$= \frac{1}{4} k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4} [k^2(k+1)^2 + 4(k+1)^3]$$

$$= \frac{1}{4} (k+1)^2 [k^2 + 4(k+1)]$$

$$= \frac{1}{4} (k+1)^2 (k \; [illegible])^2$$

[illegible]

10. (First Principle)

There are many ways [illegible] problem. Perhaps the simplest is to observe that th[illegible] n th term is simply $n^2 + n$ so that the sum S_n of this series [illegible] simply by adding the result of Exercise 1 to the sum [illegible] Example [illegible] We have then

$$S \; [illegible] \; \frac{1}{2} \; [illegible] \; \frac{1}{6} \; [illegible]$$

$$\frac{1}{6} \; [illegible]$$

$$\frac{1}{3} \; [illegible]$$

[illegible]

$$\frac{1}{3} \; [illegible]$$

You might remark as an interesting sidelight that this result also demonstrates that the product of three consecutive natural numbers is divisible by 3.

11. (First Principle)

Let A_k be the assertion that for any $k + 2$ points P_1, P_2, ..., P_{k+2} that

$S_k = m(P_1P_2) + m(P_2P_3) + \ldots + m(P_{k+1}P_{k+2}) \geq m(P_1P_{k+2})$

Initial Step:

A_1 is certainly true since it merely states the triangle inequality.

Sequential Step.

If A_k is true, ...

S_{k+1} ... $m(P_{k+2}P_{k+3})$

$\geq m(P_1P_{k+2}) + m(P_{k+2} \ldots)$

$\geq m(P_1P_{k+3})$

where ... the t...

d.

12. ...

Initial Step:

...

(1 ...) (1 ...

...

If the ab...

...

$$= (k + 1)^2 \left(1 + \frac{2k + 3}{(k+1)^2}\right)$$

$$= (k + 1)^2 + (2k + 3)$$

$$= (k^2 + 2k + 1) + (2k + 3)$$

$$= k^2 + 4k + 4$$

$$= (k + 2)^2.$$

15. (Second [illegible])

Let $U_n = n(n^2 + [illegible])$ for [illegible] natural numbers [illegible]. We note that U_n is a cubic polynomial in n. It is easy enough to see that the difference

$$V_n \quad [illegible]$$

is a quadratic [illegible] in n, and [illegible]

$$W_n = V_n \quad [illegible] = U_n \quad 2U_{n-1} + U_{n-2}$$

[illegible]

differences in this [illegible]

[illegible]

[illegible]

[illegible]

If W [illegible]

We [illegible]

[illegible]

cated [illegible]

sugge[illegible]

First we observe that $U_1 = 6$ so that the result is certainly correct in that case. Next we assume the result is true for all natural numbers less then or equal to k. We have

$$V_{k+1} = U_{k+1} - U_k = (k+1)\ [(k+1)^2 + 5] - k\,[k^2 + 5]$$
$$= [\,(k+1)^3 - k^3] + 5(k+1) - 5k$$
$$= 3k^2 + 3k + 6$$
$$= 3(k^2 + k + 2).$$

(It follows from the first principle that U_{k+1} is divisible by 3.)

Proceeding one more step we have

$$W_{k+1} = V_{k+1} - V_k = U_{k+1} - 2U_k + U_{k-1}$$
$$= 3\ \{[k^2 + k + 2] - [(k-1)^2 + (k-1) + 2]\}$$
$$= 6\ k.$$

q.e.d.

It follows that

$$U_{k+1} = 2U_k - U_{k-1} + 6k. \qquad (1)$$

Hence, since U_k and U_{k-1} are divisible by 6 we see that U_{k-1} is divisible by 6.

It is good to point out to the student that the proof is not complete at this point since the argument going from k to k + 1 must be valid for all k. In this case equation (1) is meaningless for k = 1, since U_0 is undefined (U_n is defined only for natural numbers n). There are two ways to surmount this difficulty.

a. Simply extend the interpretation of the formula for U_n so that $U_0 = 0$. The method of backward extension is often quite useful.

b. Let the k-th assertion A_k be that both U_k and U_{k+1} are divisible by 6 (rather than just U_k.) To prove the general result it then becomes necessary to establish both $U_1 = 6$ and $U_2 = 6 \cdot 3$ as special cases in the initial step. (This method is quite general. Note that it is similar to a proof of the second principle by the first.)

14. (Second Principle)

This is a problem for which the methods of proof are diverse. Most usually, the student will probably discover that the payments are in arithmetic progression beginning at the leader, going around the circle and returning to the leader again. He will then realize that all payments must be equal. Here is another approach.

Let us suppose there are n pirates in addition to the leader. We assume $n > 1$; otherwise the result is obvious. Let P_0 be the amount of payment to the leader and let $P_1, P_2, \ldots, P_n$ be the payments to the other pirates going to the right from the leader around the circle. Except for the leader, we know that each pirate receives a payment equal to the average of the two men on his right and left. It follows that for $1 \leq k \leq n - 1$ we have

(1) $$P_k = \frac{P_{k-1} + P_{k+1}}{2} \quad (k = 1, \ldots, n - 1),$$

and for $k = n$

(2) $$P_n = \frac{P_{n-1} + P_0}{2}.$$

We consider three cases that P_1 is equal to, greater or less than P_0.

a. Suppose $P_1 = P_0$. Then from

$$P_1 = \frac{P_0 + P_2}{2} = P_0$$

we have

$$P_2 = P_0.$$

Now if it is true that $P_k = P_{k-1}$ we have, following the same line of argument, that

$$P_k = \frac{P_{k+1} + P_{k-1}}{2} = P_{k-1}$$

and, therefore,

$$P_{k+1} = P_k .$$

It follows by the second principle (if for all natural numbers j less than or equal to k the values of P_k are all equal to P_0 then $P_{k+1} = P_0$) that, in so far as formula (1) holds, all values of $P_k = P_0$. In other words

$$P_k = P_0, \text{ for } k = 1, \ldots, n - 1.$$

for $k = n$ it follows from (2) that

$$P_n = \frac{P_0 + P_0}{2} = P_0.$$

We see then that if the man on the leader's right gets the same amount as the leader, so does everyone else.

b. Suppose $P_1 < P_0$. Then from

$$P_1 = \frac{P_0 + P_2}{2}$$

we have

$$\begin{aligned} P_2 &= 2P_1 - P_0 = P_1 + P_1 - P_0 \\ &< P_1 + P_0 - P_0 = P_1 \\ &< P_0 \end{aligned}$$

Assuming that $P_j < P_{j-1} < P_0$ for all natural numbers j such that $j \leq k$ we see by applying the same argument to

$$P_{k+1} = 2P_k - P_{k-1}$$

that $P_{k+1} < (P_k + P_{k-1}) - P_{k-1} < P_k < P_0$ insofar as formula (1) holds. For $k = n$ we have, in particular

$$P_n < P_{n-1} < P_0.$$

But, from (2)

$$P_0 = 2P_n - P_{n-1}$$
$$< (P_n + P_{n-1}) - P_{n-1} < P_n$$
$$< P_0,$$

a contradiction. We conclude that the man on the right cannot receive less than the leader.

c. $P_1 > P_0$. By an argument exactly parallel to that of (b) it can be shown that this is not possible.

Conclusion: Combining the results a, b, c, we see that the loot may only be divided into equal parts.

*15. (First Principle)

First we begin with the proof that p_n and q_n are relatively prime, that is, p_n and q_n have no common factor greater than one.

Initial Step: The assertion is true for $n = 1$.

Sequential Step: Suppose the assertion is true for $\frac{p_k}{q_k}$. We prove it true for $\frac{p_{k+1}}{q_{k+1}}$: From

$$p_{k+1} = p_k + 2q_k; \qquad q_{k+1} = p_k + q_k$$

we obtain

$$p_k = 2q_{k+1} - p_{k+1},$$
$$q_k = p_{k+1} - q_{k+1}$$

It follows at once that any common factor of p_{k+1} and q_{k+1} is a common factor of p_k and q_k. Since 1 is the greatest common divisor of p_{k+1} and q_{k+1}.

q.e.d.

For the purpose of answering the rest of the question we define the error at the n-th stage of approximation as

$$e_n = \frac{p_n}{q_n} - \sqrt{2}.$$

Now, let us attempt to represent l_{k+1} in terms of e_k. We have

$$e_{k+1} = \frac{p_{k+1}}{q_{k+1}} - \sqrt{2} = \frac{p_k + 2q_k}{p_k + q_k} - \sqrt{2}$$

$$= \frac{(p_k/q_k + 2}{(p_k/q_k + 1} - \sqrt{2}$$

$$= \frac{(e_k + \sqrt{2}) + 2}{(e_k + \sqrt{2}) + 1} - \sqrt{2}$$

$$= \frac{(e_k + \sqrt{2} + 2) - \sqrt{2}(e_k + \sqrt{2} + 1)}{(e_k + \sqrt{2} + 1)}$$

$$= \frac{(1 - \sqrt{2})e_k + (\sqrt{2} + 2) - (2 + \sqrt{2})}{e_k + \sqrt{2} + 1}$$

$$= \frac{(1 - \sqrt{2})\, e_k}{e_k + \sqrt{2} + 1}.$$

In order to simplify the work we multiply the numerator and denominator by 1 + 2 to obtain

(1) $$\ell_{k+1} = \frac{-e_k}{(\sqrt{2} + 1)^2 + e_k(\sqrt{2} + 1)}$$

From this result we shall obtain all we need. Two things are clear from (1). If e_k is small enough, it will have little effect in the denominator; the denominator will be positive, even greater than one, and e_{k+1} will have opposite sign from e_k.

First we observe that $|e_1| = |1 - \sqrt{2}| < 1$. If $|e_k| < 1$ we show that $|e_{k+1}| < 1$. In order to prove this we observe for the denominator D_{k+1} in (1) that

$$D_{k+1} = [(\sqrt{2} + 1)^2 + e_k(\sqrt{2} + 1)] = (\sqrt{2} + 1)[\sqrt{2} + 1 + e_k]$$
$$= (\sqrt{2} + 1)\ [\sqrt{2} + (1 + e_k)].$$

Since $1 + e_k > 0$ it follows that

(2) $$D_{k+1} > \sqrt{2}(\sqrt{2} + 1) = 2 + \sqrt{2} > 2 > 1.$$

Entering this result in (1) we see that

(3) $$|e_{k+1}| = \frac{|e_k|}{D} < |e_k| < 1.$$

We have proved by mathematical induction that $|e_n| < 1$ for all natural numbers n but we have proved more. Since the denominator D_{n+1} is positive it follows immediately from (1) that the error alternates in sign, since $e_{n+1} = -e_n/D_{n+1}$. We have now only to prove that the error can be made as small as desired. In fact, we shall prove that

$$|e_{n+1}| < \frac{1}{2^n} \qquad (n = 1, 2, 3, \ldots).$$

<u>Initial Step:</u> For $n = 1$, we have

$$|e_2| = \frac{|e_1|}{D_2} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)\,[(\sqrt{2} + 1) + (1 - \sqrt{2})]}$$

$$= \frac{\sqrt{2} - 1}{2(\sqrt{2} + 1)} = \frac{1}{2}\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) < \frac{1}{2} \cdot 1$$

$$< \frac{1}{2}.$$

<u>Sequential Step:</u>

From (2) above we have proved $D_n > 2 + \sqrt{2} > 2$ for all n. If

$$|e_{k+1}| < \frac{1}{2^k},$$

then

$$|e_{k+2}| = \frac{|e_{k+1}|}{D_{k+1}} < \frac{1}{2^k D_{k+2}} < \frac{1}{2^k\ 2}$$

$$< \frac{1}{2^{k+1}}.$$

With this result, the proof is complete.

*16. (First and Second Principles)

In solving this problem, as in many other mathematical problems, it pays to turn things around. Instead of thinking of $q(n)$ as the sum of $p(1) + p(2) + \ldots + p(n)$, we may think of $p(n)$ as the difference

$$p(n) = q(n + 1) - q(n). \qquad (1)$$

This relation suggests a converse to the theorem we wish to prove; namely, that if q is a polynomial of degree $m + 1$ then there is a polynomial of degree m for which (1) holds. We have already made use of this theorem in the solution to Exercise 13.

With this idea in mind, we are led to the following attack on the original problem: take a polynomial of degree $m + 1$ (preferably the simplest one, $u: x \to x^{m+1}$), form the differences

$$r(n) = u(n) - u(n - 1), \qquad (2)$$

and compare $r(n)$ with $p(n)$. We note that

$$\begin{aligned} r(1) &= u(1) - u(0) \\ r(2) &= u(2) - u(1) \\ &\ \ \vdots \\ r(n) &= u(n) - u(n - 1) \end{aligned}$$

and hence, adding,

$$\begin{aligned} r(1) + r(2) + \ldots + r(u) &= u(n) - u(0) \\ &= u(n) \qquad (3) \end{aligned}$$

since $u(0) = 0^{m+1} = 0$. (The result (3), though obvious, should itself be proved by mathematical induction; a proof by the first principle is easy.)

Our assertion A_k is : if p_k is a polynomial of degree k, then there exists a polynomial q of degree $k + 1$ such that

$$q(n) = p_k(1) + p_k(2) + \ldots + p_k(n)$$

for each natural number n.

Initial Step:

In this case, $p_1(x) = A + Bx$, and the sum

$$p_1(1) + p_1(2) + \ldots + p_1(n)$$

is an arithmetic progression with first term $A + B$ and common difference B.

Hence, by Exercise 2,

$$p_1(1) + p_1(2) + \ldots + p_1(n) = \frac{n}{2}[2(A + B) + (n - 1)B]$$
$$= \frac{B}{2}n^2 + (A + \frac{B}{2})n,$$

and the polynomial q such that

$$q(x) = \frac{B}{2}x^2 + (A + \frac{B}{2})x,$$

of degree 2, has the required property.

Sequential Step: We suppose that the assertions $A_1, A_2, \ldots, A_k$ are all true, that is, the result is proved for polynomials of degree at most k.

We may write any polynomial of degree k + 1 in the form

$$p_{k+1}(x) = ax^{k+1} + S_k(x) \qquad (a \neq 0)$$

where $S_k(x)$ is of degree k at most. Since, by our induction hypothesis, the sum

$$S_k(1) + S_k(2) + \ldots + S_k(n)$$

is the value at x = n of a polynomial of degree at most k + 1, we need not concern ourselves with the contribution of S_k and can devote our attention primarily to the term ax^{k+1}. We set

$$q(n) = p_{k+1}(1) + p_{k+1}(2) + \ldots + p_{k+1}(n),$$

and, using (4), we find

$$q(n) = a[1^{k+1} + 2^{k+1} + \ldots + n^{k+1}] + [S_k(1) + S_k(2) + \ldots + S_k(n)] \quad (5)$$

We wish to compare this with the sum obtained in (3). We therefore define, by analogy with (2),

$$r_m(x) = x^{m+1} - (x - 1)^{m+1}. \qquad (6)$$

If we expand $(x - 1)^{m+1}$ by the Binomial Theorem and combine terms, we obtain

$$\begin{aligned} r_m(x) &= x^{m+1} - [x^{m+1} - (m + 1)x^m + \ldots + (-1)^{m+1}] \\ &= (m + 1)x^m + t_{m-1}(x). \qquad (7) \end{aligned}$$

where t_{m-1} is a polynomial of degree $m - 1$.

Because the Binomiel Theorem also demands proof by induction, we include a special proof of (7) at the end of this discussion of Exercise 16.

We have, from (3) and (6),

$$r_m(1) + r_m(2) + \ldots + r_m(n) = n^{m+1} \qquad (8)$$

From (7),

$$x^m = \frac{1}{m + 1}[r_m(x) - t_{m-1}(x)]$$

and, setting $m = k + 1$, we get

$$x^{k+1} = \frac{1}{k + 2}[r_{k+1}(x) - t_k(x)]. \qquad (8)$$

Substituting this result in (4), we have

$$\begin{aligned} p_{k+1}(x) &= \frac{a}{k + 2}[r_{k+1}(x) - t_k(x)] + S_k(x) \\ &= br_{k+1}(x) + v_k(x) \qquad (9) \end{aligned}$$

where $b = \frac{a}{k + 2} \neq 0$ and $v_k(x) = S_k(x) - \frac{a}{k + 2} t_k(x)$ is of degree at most k. We now substitute (9) in (5) getting

$$q(a) = b[r_{k+1}(1) + r_{k+1}(2) + \ldots + r_{k+1}(n)]$$
$$+[v_k(1) + v_k(2) + \ldots + v_k(n)]$$
$$= bn^{k+2} + [v_k(1) + v_k(2) + \ldots + v_k(n)].$$

by (8). Our induction hypothesis asserts that

$$v_k(1) + v_k(2) + \ldots + v_k(n) = w_{k+1}(n),$$

where w_{k+1} is a polynomial of degree $k + 1$ at most. Hence

$$q(x) = bx^{k+2} + w_{k+1}(x)$$

is a polynomial of degree $k + 2$, and the induction is complete.

We must now prove (7)

$$r_m(x) = x^{m+1} - (x - 1)^{m+1} = (m+1)x^m + t_{m-1}(x).$$

Initial Step:

If $m = 1$, we have

$$x^2 - (x - 1)^2 = 2x - 1.$$

Sequential Step:

Assume the result is true for $m = k$. At the $(k + 1)$ th stage, we have

$$\begin{aligned} r_{k+1}(x) &= x^{k+2} - (x - 1)^{k+2} \\ &= x^{k+2} + (x - 1)[x^{k+1} - (x - 1)^{k+1}] - x^{k+1} \\ &\quad x^{k+1}[x - (x - 1)] + (x - 1)r_k(x) \\ &= x^{k+1} + (x-1)[(k+1)x^k + t_{k-1}(x)] \\ &= x^{k+1} + (k+1)x^{k+1} - (k+1)x^k + (x-1)t_{k-1}(x) \\ &= (k + 2)x^{k+1} + t_k(x) \end{aligned}$$

where $t_k(x) = -(k + 1)x^k + (x - 1)t_{k-1}(x)$ is a polynomial of degree at most k. This is the desired result.

**17.a (First Principle)

It is easy to see that we can never have $f(m) = g(n)$, since $9 = 3^2$ and consequently $g(n)$ is an even power of 3 for all natural numbers n, while $f(n)$ is always an odd power of 3. We must therefore show only that $m = n + 1$ is the least natural number m such that $f(m) > g(n)$. It is convenient to break this into two parts

a) $f(n + 1) > g(n)$, and

b) $f(r) < g(n)$ for all $r \leq n$.

We first examine a):

$$f(n + 1) > g(n) \qquad (1)$$

this means $$3^{f(n)} > 9^{g(n - 1)} = 3^{2g(n-1)} \quad (n > 1)$$

or, since $x \to 3^x$ is a strictly increasing function of x,

$$f(n) > 2g(n - 1)$$

or, since $f(n)$ and $g(n-1)$ are natural numbers

$$f(n) \geq 2g(n - 1) + 1. \qquad (2)$$

But (2) implies

$$3^{f(n)} \geq 3^{2g(n - 1) + 1} = 3 \cdot 9^{g(n - 1)}$$

or $$f(n + 1) \geq 3\, g(n) = 2g(n) + g(n)$$

or, since $g(n)$ is a natural number and therefore $g(n) \geq 1$

$$f(n + 1) \geq 2g(n) + 1. \qquad (3)$$

We have thus proved that (2) is equivalent to (1) and implies (3). This is all we need to prove a). Let our assertion A_n be $f(n + 1) > g(n)$.

Initial Step:

If $n = 1$, we have

$$f(2) = 27 \geq 2g(1) + 1 = 2 \cdot 9 + 1 = 19.$$

This implies, as we have shown

$$f(2) > g(1)$$

and A_1 is thus verified.

<u>Sequential Step</u>:

We assume the truth of A_k, which asserts that

$$f(k + 1) > g(k).$$

As we have seen, this is equivalent to

$$f(k) \geq 2g(k - 1) + 1$$

which implies

$$f(k + 1) \geq 2g(k) + 1$$

which is equivalent to

$$f(k + 2) > g(k+1).$$

Thus A_k implies A_{k+1}, and the inductive proof of part a) is complete.

We must now prove part b): $f(r) < g(n)$ for all $r \leq n$. Since, as has been observed, $x \longrightarrow 3^x$ is a strictly increasing function of x, it suffices to prove the case $r = n$.

<u>Initial Step</u>:

$f(1) = 3 < g(1) = 9.$

<u>Sequential Step</u>: Assume $f(k) < g(k)$. Then, since $g(k) \geq 1$, $f(k) < g(k) + g(k) = 2g(k)$.

But $f(k + 1) = 3^{f(k)}$, and $g(k + 1) = 9^{g(k)} = 3^{2g(k)}$, and therefore $f(k + 1) < g(k + 1)$, which completes the induction for part b) and thus the proof of the theorem.

**18. (First Principle)

(The teacher will have recognized these as Fibonacci numbers.)

Following the hint we obtain by experiment

(1) $x^n - y^n = (x + y)(x^{n-1} - y^{n-1}) - xy(x^{n-2} - y^{n-2})$

Set

(2) $x = 1 + \sqrt{5}, \qquad y = 1 - \sqrt{5}$

and

(3) $I_n = (x^n - y^n) \quad 2^n \sqrt{5}.$

Using (1) and (2) in (3) we obtain

$$In = \frac{(x + y)2^{n-1}\sqrt{5}\ I_{n-1} - xy2^{n-2}\sqrt{5}\ I_{n-2}}{2^n \sqrt{5}}.$$

From $x + y = 2$, $xy = -4$ we then have

$$In = \frac{2^n\sqrt{5}\ I_{n-1} + 2^n\sqrt{5}\ I_{n-2}}{2^n\sqrt{5}}$$

$$= \quad I_{n-1} + I_{n-2}.$$

Consequently, if I_{n-1} and I_{n-2} are integers so is I_n.

In order to frame a proof by mathematical induction we use the first principle and take for the assertion A_k that both I_k and I_{k+1} are integers.

<u>Initial Step</u>:

$$I_1 = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2\sqrt{5}} = 1,$$

$$I_2 = \frac{(6 + 2\sqrt{5}) - (6 - 2\sqrt{5})}{4\sqrt{5}} = 1.$$

<u>Sequential Step</u>:

We assume that the theorem is true for $n = k$.

We have by the argument above

$$I_{k+2} = I_{k+1} + I_k.$$

The assertion A_k which we have assumed true is that I_k and I_{k+1} are integers. Consequently their sum I_{k+2} is an integer. q.e.d.

(We could have used the second principle, pointing out that $I_{k+1} = I_k + I_{k-1}$ breaks down for $k = 1$, but that the result that I_2 is an integer holds anyway.)

Exercises 2 - 9 Solutions

1. a) $x \to 2x - 1$

   b) $x \to 2x - 1$

   c) $x \to 4x^3 - 9x^2 + 1$

   d) $x \to 4x^3 - 9x^2 + 1$

2. a) $x \to x^2$ or $x \to x^2 + 1$

   b) $x \to x^3$ or $x \to x^3 + 1$

   c) $x \to x^6 - x^3 + 8x$ or $x \to x^6 - x^3 + 8x + 1$

   d) $x \to \frac{x^5}{5} + \frac{x^3}{3} + x$ or $x \to \frac{x^5}{5} + \frac{x^3}{3} + x + 1$

3. They differ by a constant.

4. a) $\frac{x^3}{3}$ at x = [illegible]

   b) [illegible]

   c) x^4 [illegible] $\frac{x}{2}$ [illegible]

5. a) [illegible]

b) $x^3 + x$ at $x = 1$ or 2

c) $x^3 + x$ at $x = 2$ or 10

d) $10 - 2 = 8$

6. $\left[\left(16x - \frac{x^3}{3}\right) \text{ at } x = 3\right] - \left[\left(16x - \frac{x^3}{3}\right) \text{ at } x = 2\right]$

or $39 - (32 - 8/3) = 7 + 8/3 = 29/3$.

7. $\left[\left(x^4 - \frac{x^2}{2}\right) \text{ at } x = 2\right] - \left[\left(x^4 - \frac{x^2}{2}\right) \text{ at } x = 1\right]$

$= 14 - 1/2 = 13\ 1/2$.

8. a) $g(x) = 2x^3 + 2x$ or $g(x) = 2x^3 + 2x + c$

$g(2) - g(0) = 20 - 0 = 20$ or $g(2) - g(0) = (20 + c) - c$

$= 20$

b) 20

9. a) $g(x)$ [illegible]

$g(2) - g(1) =$ [illegible]

or $g(2) - g(1) =$ [illegible]

[illegible]

Since [illegible]

$g(5) - g(3)$ [illegible]

$= h(5)$ [illegible] (3) [illegible]

[illegible] Solutions

1 Eliminate a [illegible]

[illegible]

Then $\qquad -72a_3 = -33$

$$a_3 = +\frac{33}{72} = +\frac{11}{24}$$

Since $a_2 + a_3 = \frac{-13}{12}$

$$a_2 = \frac{-13}{12} - \frac{11}{24} = \frac{-37}{24}$$

$$a_1 = -3a_2 - 7a_3 - 4$$

$$= \frac{1}{24}[3.37 - 7.11 - 4.24]$$

$$= \frac{-62}{24} = \frac{-31}{12}$$

Finally, [illegible] $\frac{-1}{24}$ [illegible] $\frac{}{3}$

[illegible]

$(2, -1) \qquad \dfrac{2\,(-1)}{2} = -1$

$(4, 2) \qquad \dfrac{4(1)}{2} = 2$

3. $(-1, 2)\ (0,-1), (2,3)$

$$g_1(x) = 2\,\frac{(x-0)\,(x-2)}{(-1-0)\,(-1-2)} = \frac{2x(x-2)}{3}$$

$$g_2(x) = -1\,\frac{(x+1)\,(x-2)}{(0+1)\,(0-2)} = \frac{1}{2}\,(x+1)\,(x-2)$$

$$g_3(x) = 3\,\frac{(x+1)\,(x-0)}{(2+1)\,(2-0)} = \frac{1}{2}\,x\,(x+1)$$

$$g(x) = g_1(x) + g_2(x) + g_3(x)$$

[illegible]

Exercises 4 - 12a

1. $f(x + y) = 0$, $f(x) = 0$, $f(y) = 0$; hence (6) becomes $0 = 0 \cdot 0$. This function does not satisfy (3) because division by zero is not defined. The function f is $x \longrightarrow \frac{g(x)}{g(0)}$ and is itself not defined in the case $g(0) = 0$.
2. $f(x + y) = 1$, $f(x) = 1$, $f(y) = 1$; hence (6) becomes $1 = 1 \cdot 1$. This function also satisfies (3).
3. Equation (3) takes the form

$$\frac{Af(x+y)}{Af(x)} = \frac{Af(u+y)}{Af(u)}$$

$$\frac{f(x+y)}{f(x)} = \frac{f(u+y)}{f(u)}$$

[illegible]

f(y)

and (3) is [illegible]

[illegible]

(0) i[illegible]

re

rea

2. Since $f(x) \neq 0$, we may divide (8) by it. This gives the required equation.

Exercises 4-12d

1. $f(1/3) = a^{1/3}$ $f(1/5) = a^{1/5}$

$f(1/4) = a^{1/4}$ $f(2/5) = a^{2/5}$

$f(3/4) = a^{3/4}$ $f(3/5) = a^{3/5}$

2. $f\left(\frac{371}{1000}\right) = a^{371/1000}$

3. $f(r) = f(r \cdot 1) = [f(1)]^r = 1^r = 1$ by (13)

4. If a and r are both positive rational numbers, then so is $\frac{r}{a}$, and we have

$$f(r) = f\left(\frac{r}{a} \cdot a\right) = [f(a)]^{r/a} = 1^{r/a} = 1,$$

using (13)

Exercises 4-12e

1. [illegible]

[illegible]

[illegible]

But we just proved that

$f(2x) = [f(x)]^2$ hence

$f(3x) = [f(x)]^3$

To prove $f(mx) = [f(x)]^m$ ... integers ... real numbers ...

a) Initial step. $f(1x) = [f(x)]^1$ is obvious ...

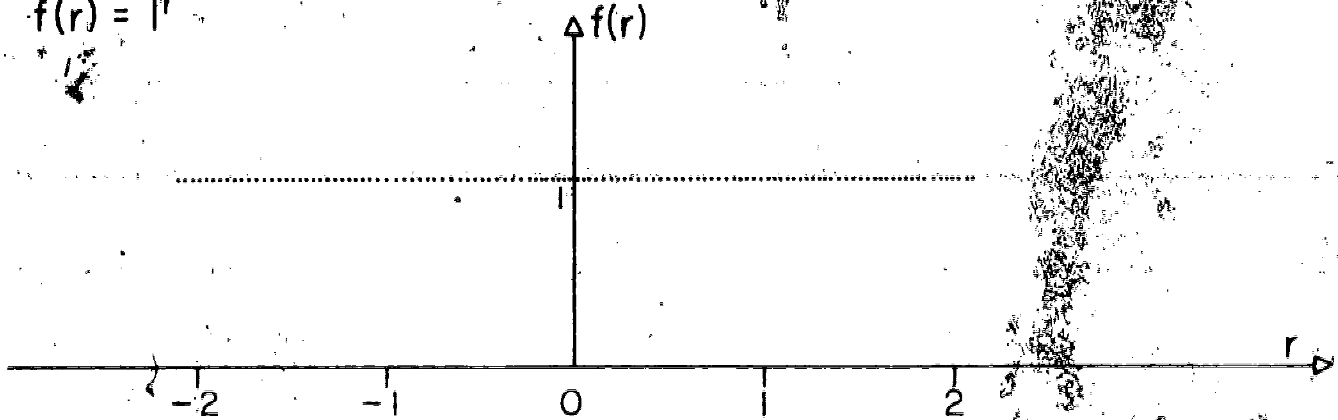
b) Sequential step. If $f(kx) = [f(x)]^k$ ... integer k. Then $f[(k+1)x] = f(kx + x)$

$= f(kx)f(x)$

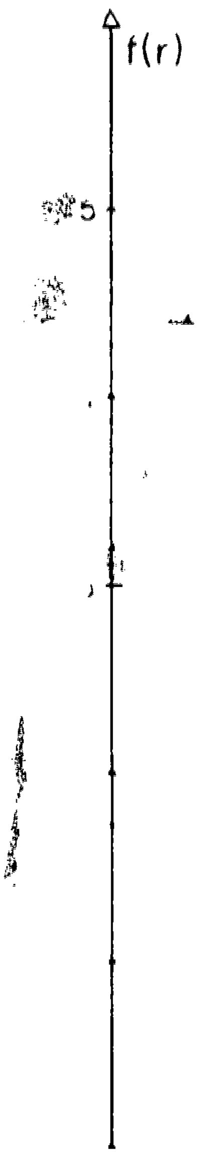
$= [f(x)]^k \cdot [f(x)]$ ... by ...

$= [f(x)]^{k+1}$

$f(r) = 1^r$



$f(r) = (\frac{1}{2})^r$



Exercises 4-12e

1. $f(1/3) = f(1/3 \cdot 1) = [f(1)]^{1/3} = a^{1/3}$

   $f(1/4) = a^{1/4}$; $f(3/4) = a^{3/4}$

   $f(1/5) = a^{1/5}$; $f(2/5) = f(2 \cdot 1/5) = [f(1/5)]^2 = a^{2/5}$;

   $f(3/5) = a^{3/5}$; $f(4/5) = a^{4/5}$

2. $f\left(\frac{371}{1000}\right) = a^{.371}$

3. If $f(1) = 1$, then $f(r) = f(r \cdot 1) = [f(1)]^r$

   $= 1^r = 1$

4. If $f(a)$ [illegible]

   [illegible]

Exercises 4-12 [illegible]

[illegible]

Similarly, $f(3x) = 3f(x)$, $f(4x) = 4f(x)$, and so on.

Proof that $f(mx) = mf(x)$

<u>Initial Step</u>: $f(1 \cdot x) = f(x) = 1 \cdot f(x)$.

<u>Sequential Step</u>: Suppose $f(kx) = kf(x)$ for some natural number k. Then

$$f[(k + 1)x] = f(kx + x) = f(kx) + f(x) \quad \text{by (1)}$$

$$= kf(x) + f(x)$$

$$= (k + 1) f(x)$$

q.e.d.

d) $f\left(m \cdot \frac{x}{m}\right) = m f\left(\frac{x}{m}\right)$

$f(x)$

Hence $f\left(\frac{x}{m}\right) = \frac{1}{m} \cdot f(x)$

e) [illegible]

$= rf(x).$

g) $f(r) = f(r \cdot 1) = rf(1)$

for all rational r by parts e) and f). But

$f(1) = a,$

hence $f(r) = ar.$

* h) If x is any irrational number, there are rational numbers r and s such that

$$r < x < s$$

and such that $x - r$ and $s - x$ are arbitrarily close to zero. We have,

$$f(r) = ar, \quad f(s) = as$$

If f is strictly increasing $a > 0$, and [illegible]

$$f(r) = ar < ax < as = f(s)$$

The difference [illegible] $ax - f(r) = ax - ar = a(x - r)$

and [illegible] $f(s) - ax = as - ax = a(s - x)$

can be made arbitrarily close to zero. [illegible] real number x and is strictly increasing [illegible] have is ax.

The argument [illegible] is similar. The required [illegible] follows.

3. [illegible]

4. [illegible]

$$\frac{3x + k}{3\,\frac{2x + 1}{3x + k}} \ [illegible]$$

$$\frac{4x}{6x} + \frac{2}{3} + 3k \ [illegible]$$

[illegible]

This must be true, in particular, for x = o (unless k = o), and therefore

$$\frac{k+2}{k^2+3} = o$$

and $\underline{k = -2}$.

Exercises 4-13

1.
$$1 = 1.000000$$
$$0.1 = 0.100000$$
$$\frac{(0.1)^2}{2!} = 0.005000$$
$$\frac{(0.1)^3}{3!} = 0.000167 \qquad e^{0.1} = 1.10517$$
$$\frac{(0.1)^4}{4!} \quad [illegible]$$
$$\frac{(0.1)^5}{5!} \quad 0.000000$$
$$1.105171$$

[illegible] affe [illegible]

2.
$$1 = 1.000000$$
$$0.01 = 0.010000$$
$$\frac{(0.01)^2}{2!} = 0.000050$$
$$\frac{(0.01)^3}{3!} \quad [illegible]$$

[illegible]

3. If [illegible] is small [illegible]

affect the first several places [illegible]

[illegible]

[illegible]

4. [illegible]

[illegible] ts the [illegible]

[illegible] when [illegible]

b)

$$1 = 1.0000 \qquad \frac{1}{2} = 0.5000$$

$$\frac{1}{4} = \underline{0.0416} \qquad \frac{1}{6} = \underline{0.0014}$$

$$1.0416 \qquad 0.5014$$

$$-\ \underline{0.5014}$$

$$0.5402$$

Ans.: 0.540

2. $0.1 = 0.1000$

$$\frac{(0.1)^3}{3!} = \underline{0.0002}$$

$$0.0998$$

Ans.: 0.100

3. a) The error, $x - \sin x$, is no greater than $\frac{x^3}{3!}$ (see Exercise 1)

We then have

$$\frac{x^3}{3!} = 0.01$$

$$x^3 = 0.06$$

[illegible]

[illegible]

b) The error is no greater than $\frac{x^2}{2!}$.

$$\frac{x^2}{2!} = 0.01$$

$$x^2 = [illegible]$$

[illegible] 0.14.

[illegible]

4. [illegible] $\frac{\sin x}{\cos x}$ [illegible] x^3 [illegible] x^2 [illegible]

[illegible] appear [illegible]

[illegible] $(-x)$ [illegible]

5. a) $e^{\frac{\pi i}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$

   b) $e^{\pi i} = \cos\pi + i\sin\pi = -1 + 0\cdot i = -1$

   c) $e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1 + 0\cdot i = -1$

   d) $e^{\frac{\pi i}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$

   e) $e^{0.5i} \approx \cos 0.5 + i\sin 0.5 = 0.8776 + 0.4794i$